

Attractor scenarios and superluminal signals in k -essence cosmology

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Cosmological scenarios with k -essence are invoked in order to explain the observed late-time acceleration of the universe. These scenarios avoid the need for fine-tuned initial conditions (the “coincidence problem”) because of the attractor-like dynamics of the k -essence field ϕ . It was recently shown that all k -essence scenarios with Lagrangians $p = L(X)\phi^{-2}$, where $X \equiv \frac{1}{2}\phi_{,\mu}\phi^{,\mu}$, necessarily involve an epoch where perturbations of ϕ propagate faster than light (the “no-go theorem”). We carry out a comprehensive study of attractor-like cosmological solutions (“trackers”) involving a k -essence scalar field ϕ and another matter component. The result of this study is a complete classification of k -essence Lagrangians that admit asymptotically stable tracking solutions, among all Lagrangians of the form $p = K(\phi)L(X)$. Using this classification, we select the class of models that describe the late-time acceleration and avoid the coincidence problem through the tracking mechanism. An analogous “no-go theorem” still holds for this class of models, indicating the existence of a superluminal epoch. In the context of k -essence cosmology, the superluminal epoch does not lead to causality violations. We discuss the implications of superluminal signal propagation for possible causality violations in Lorentz-invariant field theories.

I. INTRODUCTION AND OVERVIEW OF RESULTS

Cosmological scenarios involving a scalar field known as k -essence [1, 2, 3] are intended to explain the late-time acceleration of the universe (see Ref. [4] for a recent review of dynamical models of dark energy). An important motivation behind the k -essence scenarios is to avoid the fine-tuning of the initial conditions for the scalar field (the “coincidence problem”).

The effective Lagrangian $p(X, \phi)$ describing the dynamics of the scalar field ϕ consists of a noncanonical kinetic term,

$$p(X, \phi) = K(\phi)L(X), \quad X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi, \quad (1)$$

where $K(\phi)$ and $L(X)$ are functions determined by the underlying fundamental theory. One considers the evolution of the field ϕ coupled to gravity in a standard homogeneous cosmology in the presence of matter. With a suitable choice of the Lagrangian, the evolution of ϕ during radiation domination quickly drives the system into a region in phase space where the k -essence field ϕ has a nearly constant equation of state with $w_\phi = \frac{1}{3}$, mimicking radiation. Thus the energy density ε_ϕ of k -essence approaches a constant fraction of the energy density ε_m of the radiation. This behavior of k -essence ($w_\phi \rightarrow \text{const}$ and $\varepsilon_\phi/\varepsilon_{\text{tot}} \rightarrow \text{const}$, where $\varepsilon_{\text{tot}} \equiv \varepsilon_\phi + \varepsilon_m$) is called tracking, and the solution with $w_\phi \approx \text{const}$ is called a tracker solution.

The parameters of the Lagrangian can be adjusted such that the energy density in k -essence during the radiation era is small ($\varepsilon_m \approx \varepsilon_{\text{tot}}$), so that the standard cosmological evolution is not significantly altered. After the onset of dust domination ($w_m = 0$), the energy density in k -essence quickly becomes negligible and the evolution leaves the radiation tracker. A tracking solution

with $w_\phi = 0$ does not exist (due to a particular choice of the Lagrangian), and instead the k -essence is driven to a tracking regime with $w_\phi \approx \text{const} < 0$. Since $w_\phi < w_m$, the k -essence will eventually dominate the energy density of the dust component. The precise value of w_ϕ in that regime can be parametrically adjusted to fit the currently observed data; in particular, values $w_\phi \approx -1$ can be achieved.¹

In our terminology, a “tracking solution” is a solution for which w_ϕ approaches a fixed value, whether or not this value is equal to the equation of state parameter w_m of the dominant matter component. It is essential that the tracker solutions are stable attractors for nearby solutions. Because of this property, the field ϕ is driven into the tracker regime in the phase space with fixed values of w_ϕ and the ratio $\varepsilon_\phi/\varepsilon_{\text{tot}}$, for a wide range of initial conditions for ϕ . To construct a viable k -essence model, it is important to choose a Lagrangian $p(X, \phi)$ for which stable tracker solutions exist within the radiation- and dust-dominated cosmological eras.

Previous works concerning the dynamics of k -essence either assumed a specific form of the Lagrangian, for instance [3]

$$p(X, \phi) = \frac{L(X)}{\phi^2}, \quad (2)$$

or imposed ad hoc restrictions on the Lagrangian with the purpose of deriving exact solutions (e.g. [8]). In particular, it was assumed that $w_\phi = \text{const}$ is an exact solution of the equations of motion. However, the

¹ We note that the “phantom” values $w_\phi < -1$ cannot be reached in this single-field model; see e.g. [5, 6]. “Phantom” models such as that of Ref. [7] cannot describe the tracking behavior of k -essence since in these models $w_\phi < -1$ at all times.

physically necessary requirement is weaker: namely, one merely needs that w_ϕ should *approach* a constant value asymptotically at late times. The existence of an *exact* solution $w_\phi = \text{const}$ is not necessary. With this weaker requirement, a much wider range of Lagrangians enters the consideration.

In the present paper, we restrict our attention to Lagrangians of the “factorized” form (1) but do not impose any further a priori restrictions on the Lagrangians; neither do we require the existence of analytic exact solutions, or of solutions with $w_\phi = \text{const}$. It is only assumed that the cosmological scenario is realized with $\dot{\phi} > 0$ and that ϕ reaches arbitrarily large values. Our results can be viewed as a comprehensive extension of previous studies of attractor behavior in k -essence cosmology (e.g. [8, 9, 10]). We determine the class of Lagrangians $p(X, \phi)$ that admit stable tracking regimes in which $w_\phi \rightarrow \text{const}$, for a given value of w_m . The possible asymptotic values of w_ϕ and $\varepsilon_\phi/\varepsilon_{\text{tot}}$ are derived in each case.

The form (1) is sufficiently general to reproduce an observationally measured cosmological history [11] and covers many interesting cases, such as k -essence with purely kinetic term [12] or the “kinetic quintessence” [1]. Factorized Lagrangians have been the main focus of attention in the study of k -essence (see e.g. [13, 14, 15]). More generally, Lagrangians of the form

$$p(X, \phi) = [K_1(\phi)X^{n_1} + K_2(\phi)X^{n_2}]^{n_3}, \quad (3)$$

where n_1, n_2, n_3 are constants, can be reduced to the Lagrangian (1) by a suitable redefinition of the field ϕ . Our analysis will also apply to Lagrangians that have the asymptotic form $p \approx K(\phi)L(X)$ for $\phi \rightarrow \infty$ and for which only the large- ϕ regime is cosmologically relevant. Nonfactorizable Lagrangians, such as those studied in Refs. [8, 16, 17, 18, 19], require a separate consideration which we do not attempt here.

Recently, it was shown that the scenarios of k -essence cosmology with Lagrangians of the form (2) necessarily include an epoch when perturbations in the k -essence field propagate faster than light (the “no-go theorem” [20]). It is well known that superluminal propagation of perturbations opens the possibility of causality violations, although causality is actually preserved in many cases. This issue has been a subject of some debate, see e.g. the discussion in Refs. [21, 22, 23, 24, 25, 26, 27, 28]. One of the motivations for the present work is to determine whether the “no-go theorem,” derived for a restricted class of k -essence Lagrangians, still holds in scenarios with more general Lagrangians.

To answer this question, we performed an exhaustive analysis of all the possibilities for the existence of stable tracking solutions in ghost-free k -essence theories with positive energy density (the complete list of physical restrictions is given in Sec. II). We considered the cosmological evolution of a scalar k -essence field ϕ coupled through gravity to a matter component having a fixed equation-of state parameter w_m . In this context, we enu-

merated all Lagrangians of the form (1) that admit attractor solutions with $w_\phi \rightarrow \text{const}$ and $\varepsilon_\phi/\varepsilon_{\text{tot}} \rightarrow \text{const}$ at late times (Sec. V A). Since our task is to determine the entire class of theories admitting a certain asymptotic behavior, numerical calculations could not be used. The analytic method used for the asymptotic analysis of the dynamical evolution is outlined at the beginning of Appendix A, where all the calculations are presented in detail. This method is similar to that developed in Ref. [29] for the analysis of attractors in models of k -inflation.

Armed with the complete enumeration of stable trackers, we then selected the Lagrangians capable of providing a subdominant tracker solution during the radiation era and an asymptotically dominant tracker solution during the dust era. We show that the only appropriate class of Lagrangians consists of functions $p(X, \phi)$ of the form

$$p(X, \phi) = \frac{1 + K_0(\phi)}{\phi^2} L(X), \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0. \quad (4)$$

Since the dynamical evolution drives ϕ towards very large values, these Lagrangians are practically indistinguishable from the Lagrangians of the form (2). Then one can prove, similarly to Ref. [20], that the cosmological evolution necessarily includes an epoch where perturbations of the k -essence field ϕ propagate with a superluminal speed. Thus, we prove the “no-go theorem” starting from a much wider initial class of k -essence Lagrangians.

In Sec. III we discuss the implications of superluminal signal propagation for causality. The cosmological scenario of k -essence does not exhibit any causality violations at the classical level, despite the presence of superluminal signals. Preservation of causality in a general configuration of k -essence field can be viewed as a potential problem, on the same footing as the chronology protection problem in General Relativity [30].

II. PHYSICAL RESTRICTIONS ON LAGRANGIANS AND SOLUTIONS

In this section we consider some physically necessary restrictions on the possible Lagrangians $p(X, \phi)$ and solutions $\phi(t)$.

The main physical context for k -essence scenarios is the evolution of the k -essence field on the background of a matter component with a fixed equation of state parameter w_m . The energy density of k -essence is not necessarily dominant during this evolution. Since k -essence scenarios are proposed as an explanation of the dark energy, we do not consider the case $w_m = -1$ (during primordial inflation, one must also have $w_m > -1$ due to the necessity of the graceful exit). However, we leave open the possibility $w_m < -1$.

An important requirement for a field theory is stability. A theory for a field ϕ is stable and ghost-free if the energy density ε_ϕ is positive, the speed of sound c_s is real (not imaginary), i.e. $c_s^2 > 0$, and the Lagrangian for linear perturbations has a hyperbolic signature and a

positive sign at the kinetic term. The speed of sound for perturbations on a given background is given by [31]

$$c_s^2 = \frac{p_{,v}}{vp_{,vv}}. \quad (5)$$

To obtain the leading terms of the Lagrangian for the perturbations, one writes a perturbed solution as $\phi = \phi_0(t) + \chi(t, \mathbf{x})$ and expands the Lagrangian $p(X, \phi)$ to second order in χ ; the Lagrangian $p(X, \phi)$ is assumed to be an analytic function of X at $X = 0$. The relevant terms are those quadratic in the derivatives of χ ,

$$\begin{aligned} p(X, \phi) &= p_{,X} \frac{1}{2} \chi_{,\mu} \chi^{,\mu} + \frac{1}{2} p_{,XX} \chi_{,\mu} \chi_{,\nu} \phi_0^{,\mu} \phi_0^{,\nu} + \dots \\ &\equiv \frac{1}{2} G^{\mu\nu} \chi_{,\mu} \chi_{,\nu} + \dots \end{aligned} \quad (6)$$

It follows that linear perturbations χ propagate in the effective metric

$$G^{\mu\nu} \equiv p_{,X} g^{\mu\nu} + p_{,XX} \phi_0^{,\mu} \phi_0^{,\nu}. \quad (7)$$

The no-ghost requirement is that the metric $G^{\mu\nu}$ should have the same signature as $g^{\mu\nu}$. Regardless of whether the 4-gradient $\phi_0^{,\mu}$ is spacelike or timelike,² the resulting conditions are [32]

$$p_{,X} = \frac{1}{v} p_{,v} > 0, \quad p_{,X} + 2X p_{,XX} = p_{,vv} > 0. \quad (8)$$

In the cosmological context, the field ϕ is a function of time t only; in standard k -essence scenarios that we are presently considering, $\phi(t)$ grows monotonically with t . Hence, $\phi_0^{,\mu}$ is timelike and the velocity $v \equiv \dot{\phi}$ is positive,

$$v \equiv \frac{d\phi}{dt} = \sqrt{2X} > 0, \quad \frac{\partial}{\partial X} = \frac{1}{v} \frac{\partial}{\partial v}. \quad (9)$$

We conclude that a physically reasonable cosmological solution should satisfy (for $v > 0$) the conditions

$$vp_{,v} - p > 0, \quad p_{,v} > 0, \quad p_{,vv} > 0. \quad (10)$$

It follows that $p(v, \phi)_{v=0} \leq 0$ and that $p(v, \phi)$ is a convex, monotonically growing function of v at fixed ϕ (at least for values of ϕ and v relevant in a cosmological scenario). For factorized Lagrangians $p(v, \phi) = K(\phi)Q(v)$, we find that $Q(v)$ must be a convex, monotonically growing function of v with $Q(0) \leq 0$, and also $Q'(v) > 0$ and $Q''(v) > 0$ for all values of $v > 0$ that are relevant in a given cosmological scenario.

Finally, we assume that $K(\phi)$ has monotonic behavior at $\phi \rightarrow \infty$.

² If $\phi_0^{,\mu}$ is null, the metric $G^{\mu\nu}$ will have the correct signature if $p_{,X} > 0$ and $p_{,XX} > 0$.

III. SUPERLUMINAL SIGNALS AND CAUSALITY

One of the results of this work is a conclusion that every k -essence scenario based on attractor behavior and a Lagrangian of the form (1) will include an epoch where the perturbations of the k -essence field propagate superluminally. It is therefore pertinent to discuss the possibility of causality violations in the presence of superluminal signals.

We first consider small perturbations $\phi_0 + \delta\phi$ of an arbitrary background solution $\phi_0(x)$ in a Lorentz-invariant, nonlinear field theory. To first order, the evolution of $\delta\phi$ is described by a linear equation of the form

$$G^{\mu\nu}[\phi_0] \nabla_\mu \nabla_\nu \delta\phi + B^\mu[\phi_0] \nabla_\mu \delta\phi + C[\phi_0] \delta\phi = 0, \quad (11)$$

where the coefficients $G^{\mu\nu}$, B^μ , and C are determined by the Lagrangian and depend on the background solution ϕ_0 . Unless Eq. (11) is hyperbolic (the matrix $G^{\mu\nu}$ having signature $+- - -$ or equivalent), the theory will trivially violate causality: an initial-value (Cauchy) problem will be ill-posed in any reference frame, and the evolution of perturbations will be physically unpredictable. Therefore, it is necessary to require that $G^{\mu\nu}$ have a hyperbolic signature. Background solutions ϕ_0 that lead to a parabolic or an elliptic signature of $G^{\mu\nu}$ even in a small spacetime domain must be avoided as pathological. The cosmological solution $\phi_0(t)$ used in k -essence scenarios will be well-behaved if the conditions (10) hold. Below we assume that $G^{\mu\nu}$ has signature $+- - -$.

Within a sufficiently small spacetime domain, we may regard $G^{\mu\nu}$, B^μ , and C as constants. Then it is straightforward to derive the dispersion relation

$$G^{\mu\nu} k_\mu k_\nu + i B^\mu k_\mu + C = 0 \quad (12)$$

for plane wave perturbations $\delta\phi(x) \propto \exp[ik_\mu x^\mu]$. In order to send information (“signals” or “sounds”) by means of a perturbation $\delta\phi(x)$, one needs to create a wave front, that is, a perturbation with an extremely short wavelength and a high frequency. Thus, wave fronts propagate along wave vectors k_μ determined by the leading term in Eq. (12),

$$G^{\mu\nu} k_\mu k_\nu = 0. \quad (13)$$

Any wave packet consisting of a superposition of plane waves will propagate behind the wave front. Therefore, a 4-vector u^μ of signal velocity must lie within the sound cone,

$$G^{\mu\nu} u_\mu u_\nu > 0. \quad (14)$$

Since the “sound metric” $G^{\mu\nu}$ is determined by the local behavior of the background solution $\phi_0(x)$, the sound cone may have an arbitrary relationship with the light-cone $g^{\mu\nu} u_\mu u_\nu = 0$ determined by the spacetime metric $g^{\mu\nu}$. Thus, in some theories the sound signal worldlines

may be timelike, null, or even spacelike depending on the spatial direction of their propagation.

The speed of sound waves is therefore direction-dependent. The background tensor $G^{\mu\nu}$ determines (a class of) preferred reference frames where $G^{\mu\nu}$ is diagonal. Propagation of sound is most conveniently described in terms of sound speeds in different directions in a preferred frame. In this sense, one may say that a dynamical Lorentz violation takes place for sound waves, although the full theory (that includes the tensor $G^{\mu\nu}$ as a dynamical quantity) of course remains Lorentz-invariant.

In the context of k -essence cosmology, the sound metric $G^{\mu\nu}$ is given by Eq. (7). Preferred frames are those where the t axis coincides with the cosmological time. So in the preferred frames $\phi_0 = \phi_0(t)$ is a function of time only, and the dispersion relation is

$$\omega^2 = c_s^2 |\mathbf{k}|^2, \quad (15)$$

where \mathbf{k} is the 3-dimensional wave vector and c_s is the (direction-independent) speed of sound defined by Eq. (5). In this paper we show that the considerations of the “no-go theorem” [20] hold for those Lagrangians of the form (1) that admit scenarios of tracking k -essence. By virtue of this theorem, there exists an epoch with $c_s^2 > 1$. During this epoch (which may be quite short [20]), it is possible to send signals along spacelike worldlines.

If spacelike sound signals propagated in arbitrary *spacetime* directions, one could easily create closed worldlines made of signals, called “closed signal curves” (CSCs) in Ref. [28]. This would open a Pandora’s box of classical time travel paradoxes, also violating the unitarity of quantum theory (see e.g. [33, 34, 35, 36]). However, the allowed sound signal directions are only those within the sound cone (14). This limitation precludes the possibility of constructing CSCs within a small domain where $G^{\mu\nu} \approx \text{const}$. This can be shown as follows. Diagonalizing the tensor $G^{\mu\nu}$ within that domain, one finds a preferred reference frame $\{t, x, y, z\}$ where sound signals (whether spacelike, null, or timelike) always propagate in the positive direction along the t axis. Signals sent by conventional means also propagate in the positive t direction. Since the local coordinates $\{t, x, y, z\}$ are valid within the entire domain where $G^{\mu\nu} \approx \text{const}$, no CSCs are possible within that domain.

It is straightforward to see that no causality violations through CSCs can occur in k -essence cosmology. Since $\dot{\phi}_0 > 0$ at all times, the 4-vector $\nabla_\mu \phi_0$ is everywhere timelike and selects a global preferred reference frame. (Even if $\dot{\phi}_0 = 0$ momentarily, the preferred frame is still selected by continuity.) In this reference frame, the sound waves propagate in the direction of increasing coordinate t . Hence, there exists a global foliation of the entire spacetime by spacelike hypersurfaces of equal t . Any sound signals (whether spacelike, null, or timelike), as well as any signals sent by conventional means, will traverse these hypersurfaces in the direction of increasing t . It follows that CSCs cannot occur, either locally or globally.

Similar conclusions were reached in models of inflation having $c_s^2 > 1$ [37] as well as in situations involving a k -essence field on a black hole background [23, 26]. By itself, a superluminal speed of sound does not automatically lead to CSCs or causality violations.

In certain field theories, one can construct backgrounds $\phi_0(x)$ where CSCs are possible; a notable example is given in Ref. [22]. However, such backgrounds are artificial in the sense that they require an ad hoc configuration of the field $\phi_0(x)$. It remains to be seen whether such causality-violating backgrounds can occur as a result of the dynamical evolution of the field $\phi_0(x)$ in a cosmological context.

The problem of causality violation by CSCs is similar to the problem of closed timelike curves (CTCs) occurring in General Relativity [30]. It is difficult to find a metric $g_{\mu\nu}$ that is initially well-behaved but admits CTCs as a result of dynamical evolution (one such example is given in Ref. [38]). Hawking’s “chronology protection conjecture” states that such spacetimes containing CTCs will be always unstable due to quantum effects; but it remains an open conjecture [30]. Similar considerations apply to CSCs occurring in nonlinear field theories. It is possible that CSCs will always lead to quantum instabilities due to a similar “chronology protection” mechanism. Further work is needed to resolve this intriguing question.

IV. EQUATIONS OF MOTION

We begin by writing the well-known evolution equations for k -essence cosmology in a convenient set of variables. The equations in this section will be used at various points in the following analysis.

We consider a spatially flat FRW universe with the metric

$$g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) [dx^2 + dy^2 + dz^2], \quad (16)$$

where $a(t)$ is the scale factor. In the epoch of interest, the universe contains the dynamical k -essence field $\phi(t)$ and a matter component with energy density ε_m and pressure p_m . The matter component can be approximately treated as nondynamical in the sense that its equation of state is fixed,

$$w_m \equiv \frac{p_m}{\varepsilon_m} = \text{const}. \quad (17)$$

The energy-momentum tensor of the field ϕ is that of a perfect fluid with pressure $p(X, \phi)$ and energy density

$$\varepsilon_\phi = 2Xp_{,X} - p. \quad (18)$$

Here and below we denote partial derivatives by a comma, so $p_{,X} \equiv \partial p / \partial X$. We introduce the velocity $v \equiv \dot{\phi}$ as shown by Eq. (9). Note that the Lagrangian $p(X, \phi)$ is an analytic function of X and thus an analytic function of v^2 .

The equation of state parameter for k -essence, w_ϕ , is defined by

$$w_\phi \equiv \frac{p(X, \phi)}{\varepsilon_\phi} = \frac{p}{vp_{,v} - p}. \quad (19)$$

A factorizable Lagrangian (1) is expressed as a function of v and ϕ as follows,

$$p(X, \phi) = K(\phi)Q(v), \quad Q(v) \equiv L(X). \quad (20)$$

For a Lagrangian of this form, w_ϕ is a function of v only,

$$w_\phi(v) = \frac{Q}{vQ' - Q}, \quad (21)$$

since the energy density factorizes,

$$\varepsilon_\phi = K(\phi)\tilde{\varepsilon}_\phi(v), \quad \tilde{\varepsilon}_\phi(v) \equiv vQ'(v) - Q(v). \quad (22)$$

We assume that the functions K and Q in Eq. (20) are chosen such that $K(\phi) > 0$.

The cosmological evolution is described by the equations of motion for $\phi(t)$, $\varepsilon_m(t)$, and $a(t)$,

$$\frac{\dot{a}}{a} \equiv H = \kappa \sqrt{\varepsilon_\phi + \varepsilon_m}, \quad \kappa^2 \equiv \frac{8\pi G}{3}, \quad (23)$$

$$\frac{d}{dt}(p_{,v}(v, \phi)) \equiv \ddot{p}_{,vv} + \dot{\phi}p_{,\phi v} = -3Hp_{,v} + p_{,\phi}, \quad (24)$$

$$\dot{\varepsilon}_m = -3H(\varepsilon_m + p_m) = -3H(1 + w_m)\varepsilon_m. \quad (25)$$

The equation of motion for the field ϕ can be also rewritten as a conservation law,

$$\dot{\varepsilon}_\phi = -3H(\varepsilon_\phi + p(X, \phi)) = -3H(1 + w_\phi)\varepsilon_\phi. \quad (26)$$

The total energy density $\varepsilon_{\text{tot}} \equiv \varepsilon_\phi + \varepsilon_m$ satisfies the equation

$$\dot{\varepsilon}_{\text{tot}} = -3H\varepsilon_{\text{tot}} \left[(1 + w_m) + \frac{\varepsilon_\phi}{\varepsilon_{\text{tot}}} (w_\phi - w_m) \right]. \quad (27)$$

Since the equations of motion (23)–(25) do not depend explicitly on time, and since $\phi(t)$ is monotonic in t , we may use the value of ϕ as the time variable instead of t . Then we obtain a closed system of two first-order equations for $v(\phi)$ and $\varepsilon_m(\phi)$,

$$\frac{dv(\phi)}{d\phi} = -\frac{vp_{,v\phi} - p_{,\phi} + 3\kappa p_{,v}\sqrt{\varepsilon_m + vp_{,v} - p}}{vp_{,vv}}, \quad (28)$$

$$\frac{d\varepsilon_m}{d\phi} = -\frac{3\kappa(1 + w_m)\varepsilon_m}{v}\sqrt{\varepsilon_m + vp_{,v} - p}. \quad (29)$$

We will make extensive use of the auxiliary quantity R defined by

$$R \equiv \frac{\varepsilon_m}{\varepsilon_\phi + \varepsilon_m}. \quad (30)$$

Since energy densities ε_ϕ and ε_m are always positive, the ratio R always remains between 0 and 1. The equation of motion for $R(\phi)$ is straightforwardly derived from Eqs. (25)–(26) and can be written as

$$\frac{dR}{d\phi} = -\frac{3H}{v}R(1 - R)(w_m - w_\phi(v, \phi)). \quad (31)$$

We may reformulate the equations of motion (28)–(29) as a closed system of equations involving only the variables $v(\phi)$ and $R(\phi)$. Since

$$\varepsilon_\phi + \varepsilon_m = \frac{\varepsilon_\phi}{1 - R} = \frac{vp_{,v} - p}{1 - R}, \quad (32)$$

we obtain

$$\frac{dv}{d\phi} = -\frac{1}{vp_{,vv}} \left[vp_{,v\phi} - p_{,\phi} + 3\kappa p_{,v}\sqrt{\frac{vp_{,v} - p}{1 - R}} \right], \quad (33)$$

$$\frac{dR}{d\phi} = -\frac{3\kappa}{v}R\sqrt{1 - R}\sqrt{vp_{,v} - p} \left(w_m - \frac{p}{vp_{,v} - p} \right). \quad (34)$$

For Lagrangians of the form (20), these equations are rewritten as

$$\frac{dv}{d\phi} = -c_s^2(v) \left[\frac{(\ln K)_{,\phi} v}{1 + w_\phi(v)} + 3\kappa \sqrt{\frac{K(\phi)\tilde{\varepsilon}_\phi(v)}{1 - R}} \right], \quad (35)$$

$$\frac{dR}{d\phi} = -\frac{3\kappa}{v}R\sqrt{1 - R}\sqrt{K(\phi)\tilde{\varepsilon}_\phi(v)}(w_m - w_\phi(v)). \quad (36)$$

Here $\tilde{\varepsilon}_\phi(v)$, $c_s^2(v)$, and $w_\phi(v)$ are understood as fixed functions of v ,

$$\tilde{\varepsilon}_\phi(v) \equiv vQ' - Q, \quad c_s^2(v) \equiv \frac{Q'}{vQ''}, \quad w_\phi(v) \equiv \frac{Q(v)}{\tilde{\varepsilon}_\phi(v)}, \quad (37)$$

determined by the given Lagrangian $p(v, \phi) = Q(v)K(\phi)$. These functions satisfy the following equations,

$$\frac{d}{dv}\tilde{\varepsilon}_\phi(v) = \frac{1 + w_\phi(v)}{vc_s^2(v)}\tilde{\varepsilon}_\phi(v), \quad (38)$$

$$\frac{d}{dv}w_\phi(v) = \frac{1 + w_\phi(v)}{v} \left[1 - \frac{w_\phi(v)}{c_s^2(v)} \right]. \quad (39)$$

V. VIABLE LAGRANGIANS FOR TRACKING SOLUTIONS

The detailed analysis of asymptotically stable solutions is given in Appendix A. Each asymptotically stable solution is characterized by the asymptotic values of $v = \dot{\phi}$ and $R = \varepsilon_m/\varepsilon_{\text{tot}}$, considered as functions of ϕ :

$$v_0 \equiv \lim_{\phi \rightarrow \infty} v(\phi), \quad R_0 \equiv \lim_{\phi \rightarrow \infty} \frac{\varepsilon_m(\phi)}{\varepsilon_{\text{tot}}(\phi)}. \quad (40)$$

As a summary of the results, we list all of the possibilities, together with the requirements on the Lagrangian

$p = K(\phi)Q(v)$ and the allowed values of v_0 , R_0 , w_m , and $w_\phi(v_0)$. [Note that the function $K(\phi)$ can be always multiplied by a constant, to be absorbed in $Q(v)$.] The requirements listed are necessary and sufficient conditions for the asymptotic stability of tracker solutions. The applicability of these tracker scenarios to k -essence cosmology is analyzed in subsections V B, V C, and V D.

A. Tracker solutions

Case 1. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0. \quad (41)$$

The value v_0 is determined from $w_\phi(v_0) = w_m$, and then R_0 is given by

$$R_0 = 1 - \frac{9\kappa^2 Q'(v_0)^2}{4\tilde{\varepsilon}_\phi(v_0)}. \quad (42)$$

This value of R_0 must satisfy $0 < R_0 < 1$ (the possibility $R_0 = 0$ is equivalent to case 2). The conditions

$$v_0 \neq 0, \quad c_s^2(v_0) \neq 0, \quad |w_m| < 1, \quad w_m < c_s^2(v_0), \quad \tilde{\varepsilon}_\phi(v_0) \neq 0 \quad (43)$$

must hold.

Case 2. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0. \quad (44)$$

The value v_0 is determined from

$$3\kappa\sqrt{\tilde{\varepsilon}_\phi(v_0)} = \frac{2v_0}{1 + w_\phi(v_0)} \quad (45)$$

and must satisfy $v_0 \neq 0$. The following conditions must hold,

$$w_\phi(v_0) < w_m, \quad |w_\phi(v_0)| < 1, \quad c_s^2(v_0) \neq 0. \quad (46)$$

The tracker solution has $R_0 = 0$ (k -essence dominates at late times).

Case 3. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^\alpha}, \quad \lim_{\phi \rightarrow \infty} \frac{\ln K_0(\phi)}{\ln \phi} = 0, \quad (47)$$

i.e. the function K_0 either tends to a constant, or grows or decays slower than any power of ϕ at $\phi \rightarrow \infty$. This condition determines the value of α . This value of α must satisfy

$$2 < \alpha < 1 + \frac{2}{1 + w_m}. \quad (48)$$

The interval for α is nonempty if

$$|w_m| < 1. \quad (49)$$

The value of α determines v_0 by

$$\alpha = 2 \frac{1 + w_\phi(v_0)}{1 + w_m}. \quad (50)$$

The resulting value of v_0 must satisfy the conditions

$$v_0 \neq 0, \quad c_s^2(v_0) > w_\phi(v_0) > w_m, \quad c_s^2(v_0) \neq 0, \quad \tilde{\varepsilon}_\phi(v_0) \neq 0. \quad (51)$$

The tracker solution has $R_0 = 1$ (k -essence is negligible).

Case 4. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad (52)$$

where the function K_0 must satisfy

$$\lim_{\phi \rightarrow \infty} K_0(\phi) = 0, \quad \lim_{\phi \rightarrow \infty} \frac{\ln K_0(\phi)}{\ln \phi} = 0, \quad (53)$$

i.e. $K_0(\phi)$ decays slower than any power of ϕ at $\phi \rightarrow \infty$. The value of v_0 is determined from the conditions

$$w_\phi(v_0) = w_m, \quad |w_m| < 1. \quad (54)$$

The following conditions must then hold,

$$v_0 \neq 0, \quad c_s^2(v_0) > w_m, \quad \tilde{\varepsilon}_\phi(v_0) \neq 0, \quad c_s^2(v_0) \neq 0. \quad (55)$$

The tracker solution has $R_0 = 1$ (k -essence is negligible).

Case 5. The function $K(\phi)$ decays slower than $\phi^{-\alpha}$ (or grows), where

$$\alpha \equiv \frac{2}{1 + w_m}, \quad -1 < w_m < 0. \quad (56)$$

More precisely,

$$K(\phi) = \frac{K_0(\phi)}{\phi^\alpha}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty. \quad (57)$$

The value of v_0 is determined as a root of $Q(v_0) = 0$ and $Q'(v_0) = 0$, i.e. we must have a Taylor expansion near $v = v_0$ of the form

$$Q(v) = \frac{Q_0}{nv_0}(v - v_0)^n, \quad n \geq 2, \quad Q_0 > 0. \quad (58)$$

Then the tracker solution has $R_0 = 1$ (k -essence is negligible) and $w_\phi(v_0) = 0$. The value of v must be above v_0 at all times (or else $c_s^2 < 0$).

Case 6. The function $Q(v)$ has an expansion at $v = 0$ of the form

$$Q(v) = Q_1 v^n + o(v^n), \quad Q_1 > 0, \quad n > 2. \quad (59)$$

This determines the value of n . The function $K(\phi)$ decays slower than $\phi^{-\alpha}$ (or grows), where

$$\alpha \equiv \frac{2n}{(n-1)(1+w_m)}. \quad (60)$$

More precisely,

$$K(\phi) = \frac{K_0(\phi)}{\phi^\alpha}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty. \quad (61)$$

The condition

$$-\frac{n-3}{n-1} < w_m < \frac{1}{n-1} \quad (62)$$

must hold. Then the tracker solution has $R_0 = 1$ (k -essence is negligible), $v_0 = 0$, $w_\phi(v_0) = \frac{1}{n-1}$, and $c_s^2(v_0) = \frac{1}{n-1}$.

Case 7. The function $K(\phi)$ has the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad (63)$$

where the function $K_0(\phi)$ is such that

$$\lim_{\phi \rightarrow \infty} K_0(\phi) > \frac{1}{9\kappa^2 Q_1} \quad \text{or} \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty. \quad (64)$$

The function $Q(v)$ has an expansion at $v = 0$ of the form

$$Q(v) = Q_1 v^2 + o(v^2), \quad Q_1 > 0. \quad (65)$$

We must have $w_m > 1$. The tracker solution has $R_0 = 0$ (k -essence dominates), $v_0 = 0$, and $w_\phi(v_0) = c_s^2(v_0) = 1$.

Case 8. The function $K(\phi)$ decays slower than ϕ^{-2} or grows,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty. \quad (66)$$

The value of v_0 is determined from $Q'(v_0) = 0$, $Q(v_0) < 0$. More precisely, we have an expansion near $v = v_0$,

$$Q(v) = Q_0 + Q_2(v - v_0)^n, \quad Q_0 < 0, \quad n \geq 2. \quad (67)$$

We must have $v_0 \neq 0$ and $w_m > -1$. The tracker solution has $R_0 = 0$ (k -essence dominates) and $w_\phi(v_0) = -1$. The value of v must be above v_0 at all times (or else $c_s^2 < 0$).

Case 9. The function $K(\phi)$ decays slower than ϕ^{-2} or grows,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty. \quad (68)$$

The value of v_0 is determined from $Q'(v_0) = 0$, $Q(v_0) = 0$. More precisely, we have an expansion near $v = v_0$,

$$Q(v) = Q_1(v - v_0)^n, \quad n \geq 2. \quad (69)$$

We must have $w_m > 0$ and $v_0 \neq 0$. The tracker solution has $R_0 = 0$ (k -essence dominates) and $w_\phi(v_0) = 0$. The value of v must be above v_0 at all times (or else $c_s^2 < 0$).

Case 10. The function $K(\phi)$ decays slower than ϕ^{-2} or grows,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty. \quad (70)$$

The function $Q(v)$ must have an expansion near $v = 0$ of the form

$$Q(v) = -Q_0 + Q_1 v^n, \quad Q_0 > 0, \quad n \geq 2. \quad (71)$$

We must have $w_m > -1$. The tracker solution has $R_0 = 0$ (k -essence dominates), $v_0 = 0$, and $w_\phi(v_0) = -1$.

Case 11. The function $K(\phi)$ decays slower than ϕ^{-2} or grows,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty. \quad (72)$$

The function $Q(v)$ must have an expansion near $v = 0$ of the form

$$Q(v) = Q_1 v^n + o(v^n), \quad Q_1 > 0, \quad n > 2. \quad (73)$$

This determines the value of n . The condition

$$w_m > \frac{1}{n-1} \quad (74)$$

must hold. The tracker solution has $R_0 = 0$ (k -essence dominates), $v_0 = 0$, and $w_\phi(v_0) = c_s^2 = \frac{1}{n-1}$.

Case 12. The function $Q(v)$ must have an expansion near $v = 0$ of the form

$$Q(v) = Q_1 v^n + Q_2 v^{n+p}, \quad Q_1 > 0, \quad n > 2, \quad p > 0. \quad (75)$$

This determines the values of n and p . The function $K(\phi)$ must be of the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad (76)$$

where $K_0(\phi)$ must satisfy

$$\lim_{\phi \rightarrow \infty} K_0(\phi) = \infty, \quad \int^\infty \frac{d\phi}{\phi} K_0^{-\frac{p}{n-2}}(\phi) = \infty. \quad (77)$$

(The function $K_0(\phi)$ grows slower than $(\ln \phi)^{(n-2)/p}$.) We must have $w_m = \frac{1}{n-1}$. The tracker solution has $R_0 = 0$ (k -essence dominates), $v_0 = 0$, and $w_\phi(v_0) = c_s^2 = \frac{1}{n-1}$.

B. Radiation-dominated era

We now select Lagrangians that admit tracker solutions during radiation domination, $w_m = \frac{1}{3}$. In order not to violate the nucleosynthesis bound, the energy density of k -essence must be subdominant throughout the radiation era [3],

$$R_0 \gtrsim 0.99. \quad (78)$$

Admissible trackers may have a value R_0 within the range $0.99 \lesssim R_0 < 1$, or $R_0 = 1$. A solution with $0 < R_0 < 1$ is only possible with Lagrangians given by case 1,

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0. \quad (79)$$

We denote by v_r the asymptotic value of v during the radiation era. Possible values of v_r are determined from $w_\phi(v_r) = \frac{1}{3}$, and v_r must satisfy

$$c_s^2(v_r) > \frac{1}{3}, \quad \tilde{\varepsilon}_\phi(v_r) \neq 0, \quad v_r \neq 0. \quad (80)$$

The corresponding value of R_0 must respect the bound (78),

$$R_0 = 1 - \frac{9\kappa^2 Q'^2}{4\tilde{\varepsilon}_\phi} \Big|_{v=v_r} \gtrsim 0.99. \quad (81)$$

Solutions with $R_0 = 1$ and $w_m = \frac{1}{3}$ are possible in cases 3, 4, and 6. The first set of solutions is given by

$$K(\phi) = \frac{K_0(\phi)}{\phi^\alpha}, \quad \lim_{\phi \rightarrow \infty} \frac{\ln K_0(\phi)}{\ln \phi} = 0, \quad (82)$$

where $2 < \alpha < \frac{5}{2}$. Admissible functions $K_0(\phi)$ decay or grow slower than any power of ϕ , e.g. $K_0(\phi) \propto (\ln \phi)^\beta$. Admissible values of v_r are determined from the conditions

$$w_\phi(v_r) = \frac{2\alpha}{3} - 1, \quad \tilde{\varepsilon}_\phi(v_r) \neq 0, \quad v_r \neq 0. \quad (83)$$

The second set of Lagrangians is

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0, \quad \lim_{\phi \rightarrow \infty} \frac{\ln K_0(\phi)}{\ln \phi} = 0. \quad (84)$$

The possible values of v_r are determined from $w_\phi(v_r) = \frac{1}{3}$, and the following conditions must be also satisfied,

$$c_s^2(v_r) > \frac{1}{3}, \quad \tilde{\varepsilon}_\phi(v_r) \neq 0, \quad v_r \neq 0. \quad (85)$$

The third set of admissible Lagrangians is described by case 6 with $n = 3$, namely

$$K(\phi) = \frac{K_0(\phi)}{\phi^{9/4}}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty, \quad (86)$$

$$Q(v) = Q_1 v^3 + o(v^3), \quad Q_1 > 0. \quad (87)$$

In this case, $v_r = 0$. The solution of case 6 with $n \geq 4$ cannot be used since the condition (62) cannot be satisfied with $w_m = \frac{1}{3}$.

C. Dust-dominated era

We now select the tracker solutions that exist for $w_m = 0$. In order to describe the late-time domination of k -essence, we must look for solutions with $w_\phi < -\frac{1}{3}$ and $R_0 = 0$. The possible trackers are cases 2, 8, and 10.

In case 2, the Lagrangian must satisfy

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0. \quad (88)$$

We denote by v_d the asymptotic value of v during the dust era. The admissible values of $v_d \neq 0$ are determined from

$$3\kappa\sqrt{\tilde{\varepsilon}_\phi(v_d)} = \frac{2v_d}{1 + w_\phi(v_d)}. \quad (89)$$

In addition, the following conditions must be satisfied:

$$-1 < w_\phi(v_d) < 0, \quad c_s^2(v_d) \neq 0. \quad (90)$$

The second set of Lagrangians is for cases 8 and 10,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty. \quad (91)$$

This condition for $K(\phi)$ is satisfied, for example, by $K(\phi) \propto \phi^\alpha$ with $\alpha > -2$. The value v_d must be such that

$$Q(v_d) < 0, \quad Q'(v_d) = 0, \quad (92)$$

while we may have either $v_d \neq 0$ or $v_d = 0$.

D. Viable scenarios

Having listed all the Lagrangians that admit desired solutions in the radiation- and dust-dominated eras, it remains to determine the overlap between these classes of Lagrangians. By comparing the requirements on the functions $K(\phi)$ and $Q(v)$, we find only two possibilities for trackers in the radiation/dust era: case 1/case 2 and case 6/case 8.

The first set of Lagrangians (case 1/case 2) is

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0. \quad (93)$$

In the radiation era, the asymptotic value of v is given by $v_r \neq 0$ such that

$$w_\phi(v_r) = \frac{1}{3}, \quad c_s^2(v_r) > \frac{1}{3}, \quad \tilde{\varepsilon}_\phi(v_r) \neq 0, \quad (94)$$

and the dust attractor is given by $v_d \neq 0$ such that Eqs. (89)–(90) hold. These Lagrangians describe the well-known scenario [2] where the k -essence tracks radiation during the radiation era and eventually starts to dominate in the dust era. The function $Q(v)$ must be chosen to satisfy the conditions of cases 1 and 2. Additionally, one must exclude the possibility of a dust tracker (case 1, $w_m = 0$) by adjusting $Q(v)$ such that the conditions of case 1 are not satisfied for $w_\phi(v_0) = w_m = 0$ [3].

The second set of Lagrangians is described by case 6/case 8. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = \infty.$$

The function $Q(v)$ must be such that

$$Q(v) = Q_1 v^3 + o(v^3), \quad Q_1 > 0. \quad (95)$$

Then the asymptotic values of v are $v_r = 0$ in the radiation era (where $w_\phi \approx \frac{1}{2}$) and $v_d \neq 0$ in the dust era (where $w_\phi \approx -1$). The value v_d must be a root of $Q'(v)$ such that

$$Q(v_d) < 0, \quad Q'(v_d) = 0. \quad (96)$$

This scenario, however, has a fatal flaw. The attractor of case 8 requires that $v > v_d$ at all times, while the attractor of case 6 is realized at very small $v \approx 0$. Therefore, a transition from the first attractor to the second will necessarily involve values of $v < v_d$ for which the theory is unstable since $c_s^2(v) < 0$. Hence, this scenario must be discarded.

Thus we conclude that successful models of k -essence are produced only by Lagrangians described by Eq. (93) under the conditions of case 1 and case 2.

E. The existence of a superluminal epoch

We have shown that the only viable k -essence scenario is described by case 1/case 2 of Sec. V A. Now we demonstrate that in these scenarios $c_s^2(v_*) > 1$ for some value v_* that is reached during the dust-dominated epoch. The argument is similar to that in Ref. [20].

Since $Q(v_r) > 0$ and $Q(v_d) < 0$, while $Q(v)$ is a monotonically growing function of v , we must have $v_d < v_r$. In both scenarios of case 1 and case 2, the asymptotic fraction of the energy density R_0 is equal to a certain function F of v_0 ,

$$R_0 = F(v_0) \equiv 1 - \frac{9\kappa^2}{4} \frac{Q'^2}{vQ' - Q} \Big|_{v=v_0}. \quad (97)$$

In case 2, $F(v_0) = 0$ due to Eq. (45); therefore, we may describe both cases 1 and 2 by a single function $F(v_0)$. We note that $\tilde{\epsilon}_\phi(v) = vQ'(v) - Q(v)$ is a monotonically growing function of v because

$$\frac{d}{dv} \tilde{\epsilon}_\phi(v) = vQ''(v) > 0 \quad \text{for } v > 0. \quad (98)$$

Since $\tilde{\epsilon}_\phi(v) > 0$ for every relevant value of v , it follows that $F(v)$ is a continuous function for these v . For a successful model of k -essence, the radiation tracker must have $F(v_r) \gtrsim 0.99$ and the dust tracker must have $F(v_d) = 0$. During the evolution from the first tracker to the second, the value of v must traverse the interval $[v_d, v_r]$. The condition $F(v_d) < F(v_r)$ implies (due to the continuity of F) that there exists a value $v_1 \in [v_d, v_r]$ such that $F'(v_1)$ is positive:

$$F'(v_1) = - \frac{9\kappa^2 Q' Q''}{2(vQ' - Q)^2} \left[\frac{vQ'}{2} - Q \right] \Big|_{v=v_1} > 0. \quad (99)$$

Since $Q' > 0$, $Q'' > 0$, and $\tilde{\epsilon}_\phi = vQ' - Q > 0$ for all $v \in [v_d, v_r]$, we can simplify this condition to

$$\frac{vQ'}{2} - Q \Big|_{v=v_1} < 0, \quad (100)$$

or equivalently to

$$w_\phi(v_1) = \frac{Q}{vQ' - Q} \Big|_{v=v_1} > 1. \quad (101)$$

The equation of state parameter $w_\phi(v)$ is a continuous function of v that satisfies

$$0 = w_\phi(v_d) < 1 < w_\phi(v_1). \quad (102)$$

Hence, there exists a value $v_* \in [v_d, v_1]$ such that $w_\phi(v_*) > 1$ and $w'_\phi(v_*) > 0$.

Finally, we show that $c_s^2(v_*) > 1$ follows from the conditions $w_\phi(v_*) > 1$ and $w'_\phi(v_*) > 0$. According to Eq. (39), we have

$$w'_\phi(v) = \frac{(1 + w_\phi)(c_s^2 - w_\phi)}{vc_s^2} \Big|_{v=v_*} > 0. \quad (103)$$

Therefore

$$c_s^2(v_*) > w_\phi(v_*) > 1. \quad (104)$$

Since $c_s^2(v)$ is a continuous function, this demonstrates the existence of an interval of values of v within $[v_d, v_r]$ where $c_s^2(v) > 1$. This superluminal epoch occurs during the dust-dominated era.

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Appendix A: ASYMPTOTICALLY STABLE SOLUTIONS

The standard analysis of the k -essence trackers (e.g. [3]) involves several simplifying but restrictive assumptions concerning the behavior of the solutions. A wider range of k -essence models will be obtained if some of these assumptions are lifted. Let us therefore characterize the desired features of the cosmological evolution of k -essence in a general manner.

Scenarios of k -essence are based on the assumption that the field ϕ has an almost constant equation of state parameter (w_ϕ) during a cosmologically long epoch while another matter component dominates the energy density of the universe. Eventually, the k -essence itself becomes dominant and plays the role of ‘‘dark energy,’’ again with an approximately constant w_ϕ . It is important that the solution curves serve as attractors for all neighbor solutions. In that case, the value of w_ϕ at late times is essentially independent of the initial conditions.

When the radiation-dominated epoch gives way to the epoch of dust domination, the behavior of k -essence will change in a model-dependent way. However, it is technically convenient to study the behavior of k -essence under the assumption that the dominant matter component has

a fixed equation of state for all time. Then the existence of tracker solutions will be found by studying the asymptotic behavior of the solutions at $t \rightarrow \infty$ (equivalently, at $\phi \rightarrow \infty$). This is the approach taken in this paper.

The evolution of k -essence together with a single matter component is described by the equations of motion (EOM) shown above as Eqs. (28)–(29) in terms of the variables $\{v(\phi), \varepsilon_m(\phi)\}$. We call a solution $\{v(\phi), \varepsilon_m(\phi)\}$ *asymptotically stable* if $w_\phi(\phi)$ tends to a constant at $\phi \rightarrow \infty$ and if *all* neighbor solutions (at least within a finite domain of attraction) also approach the same value of w_ϕ . In this section, we restrict our attention to asymptotically stable solutions with one matter component. Since reasonable values of w_ϕ are within the interval $[-1, 1]$, it is justifiable to ignore solutions where w_ϕ tends to infinity at late times. In principle, one could also have solutions where $w_\phi(\phi)$ oscillates without reaching any limit as $\phi \rightarrow \infty$, but such solutions are of little physical interest since the value of w_ϕ at the end of a given cosmological epoch will then be largely unpredictable. Since in the models under consideration w_ϕ is a function of v only, solutions $v(\phi)$ that oscillate without reaching any limit are also excluded. Applicability of the effective field theory requires that the derivatives of ϕ remain bounded; thus $v = \dot{\phi}$ cannot diverge to infinity as $\phi \rightarrow \infty$ and must also tend to a constant value, $v(\phi) \rightarrow v_0 < \infty$.

There may also exist solutions with initially negligible but growing ratio $\varepsilon_\phi/\varepsilon_m$. Such solutions may have a stable behavior with an almost constant w_ϕ for a finite (but very long) time, until the energy density of k -essence starts to dominate. We do not consider such “transient attractors” in the present paper.

Our main task is to deduce the possible k -essence Lagrangians $p(X, \phi)$ that admit physically meaningful asymptotically stable solutions. We consider only Lagrangians that have a factorized form (20).³ We assume that the matter component has a constant equation of state parameter w_m such that $w_m \neq -1$.

It will be convenient to use also the auxiliary variable $R(\phi)$ satisfying the EOM (36). Since the values of R are limited to the interval $[0, 1]$, any asymptotically stable solution will necessarily approach a constant value, $R(\phi) \rightarrow R_0$ as $\phi \rightarrow \infty$. The possible values of R_0 and v_0 are yet to be determined; the cases when R_0 or v_0 assume critical values ($R_0 = 0$, $R_0 = 1$, $v_0 = 0$) will need to be treated separately.

The general method of analysis is the following. We have a system of nonlinear EOM parameterized by a pair of functions $K(\phi), Q(v)$; the general solution of the EOM is not available in closed form. Our purpose is

to determine the functions $K(\phi), Q(v)$ for which a solution of the EOM exists with the asymptotic stability property. We first *assume* the existence of an asymptotically stable solution $\{v(\phi), \varepsilon_m(\phi)\}$ and derive the *necessary* conditions on the functions $K(\phi)$ and $Q(v)$ that admit such solutions (perhaps in more convenient variables, such as $\{v(\phi), R(\phi)\}$). At this step, there will be many cases corresponding to different asymptotic behavior of $v(\phi)$ and $R(\phi)$. For instance, $v(\phi)$ may tend either to a nonzero constant or to zero, etc. In each case, we then obtain the *general* solution of the EOM (with two integration constants) near the assumed stable solution (e.g. $v(\phi) = v_0 - A(\phi)$, with $A(\phi)$ very small). At this point, it is possible to make simplifying assumptions because we only consider the solutions in the asymptotic limit $\phi \rightarrow \infty$ and infinitesimally close to an assumed trajectory. We then investigate whether the general solution is attracted to the assumed stable solution. In this way, we either obtain *sufficient* conditions for the existence of a stable solution of an assumed type, or conclude that no stable solution exists in a given case. After enumerating all the cases, we will thus obtain necessary and sufficient conditions on $K(\phi)$ and $Q(v)$ for every possible type of stable tracking behavior.

Let us begin by drawing some general consequences about the asymptotic behavior of stable solutions at $\phi \rightarrow \infty$. Rewriting Eq. (29) as

$$\frac{d\varepsilon_m^{-1/2}(\phi)}{d\phi} = \frac{3\kappa(1+w_m)}{2v(\phi)\sqrt{R(\phi)}}, \quad (\text{A1})$$

and noting that the right-hand side of Eq. (A1) is bounded away from zero, we conclude that $\varepsilon_m(\phi)$ decays either as ϕ^{-2} or faster at $\phi \rightarrow \infty$, depending on whether $v(\phi)\sqrt{R(\phi)}$ tends to zero at large ϕ . In the following subsections, we consider all the possible cases.

Based on the motivation for introducing k -essence, we have assumed that $w_m \neq -1$. According to Eq. (A1), for $w_m < -1$ (phantom matter) the energy density ε_m will satisfy the differential inequality

$$\frac{d}{dt}\varepsilon_m^{-1/2} = -\frac{3\kappa|1+w_m|}{2\sqrt{R(\phi)}} < -C_1, \quad (\text{A2})$$

where C_1 is a positive constant. Thus, $\varepsilon_m(t)$ will reach infinity in finite time regardless of the behavior of $R(\phi)$ and $v(\phi)$. However, this time can be quite long and the phantom behavior might be only a temporary phenomenon. Therefore, we will use the property $w_m \neq -1$ but avoid assuming that $w_m > -1$.

In the analysis below, we will also use the following elementary facts:

a) If a function $F(\phi)$ monotonically goes to a constant at $\phi \rightarrow \infty$, then $F'(\phi)$ decays faster than ϕ^{-1} . This is easily established using the identity

$$F(0) - \lim_{\phi \rightarrow \infty} F(\phi) = - \int_0^\infty F'(\phi) d\phi < \infty, \quad (\text{A3})$$

³ Since the analysis uses only the properties of the Lagrangian in the asymptotic limit $\phi \rightarrow \infty$, our results will apply to more general Lagrangians that have the form $p = K(\phi)L(X)$ asymptotically at large ϕ and fixed X .

which means that $F'(\phi)$ is integrable at $\phi \rightarrow \infty$. Hence, $F'(\phi)$ decays faster than ϕ^{-1} at $\phi \rightarrow \infty$.

b) If a function $F(\phi)$ is monotonic, then $F'(\phi) \rightarrow 0$ if and only if

$$\lim_{\phi \rightarrow \infty} \frac{F(\phi)}{\phi} = 0. \quad (\text{A4})$$

This statement follows from the L'Hopital's rule in case $F(\phi) \rightarrow \infty$, and is trivial in case $F(\phi)$ has a finite limit at $\phi \rightarrow \infty$.

1. Energy density $\varepsilon_m \propto \phi^{-2}$ and $R_0 \neq 1$, main case

According to Eq. (A1), the asymptotic behavior $\varepsilon_m \propto \phi^{-2}$ is possible only if $v(\phi)\sqrt{R(\phi)}$ stays bounded away from zero as $\phi \rightarrow \infty$, in other words if $v_0 \neq 0$ and $R_0 \neq 0$. We also assume $R_0 \neq 1$, meaning that the energy density of k -essence tracks the matter component; thus $\varepsilon_\phi(\phi) \propto \phi^{-2}$ as well. It follows that $H(\phi) \propto \sqrt{\varepsilon_\phi(\phi)} \propto \phi^{-1}$, and then Eq. (31) yields

$$\frac{dR(\phi)}{d\phi} \propto \frac{w_m - w_\phi(\phi)}{\phi}. \quad (\text{A5})$$

Since $R(\phi) \rightarrow \text{const}$, the derivative $dR/d\phi$ must decay faster than ϕ^{-1} as $\phi \rightarrow \infty$. Hence $w_\phi(\phi) \rightarrow w_m$ as $\phi \rightarrow \infty$. This is the standard tracker behavior: the equation of state parameters of k -essence and matter become almost equal at late times.

By assumption, at large ϕ the Lagrangian is factorized, $p = K(\phi)Q(v)$, and then we have

$$w_m = w_\phi(v_0) = \frac{Q(v_0)}{v_0 Q'(v_0) - Q(v_0)}. \quad (\text{A6})$$

This algebraic equation determines the possible values of v_0 for a given w_m . (Tracker solutions of this type are impossible if this equation has no roots.) The property $\varepsilon_\phi(\phi) \propto \phi^{-2}$ becomes

$$\varepsilon_\phi(\phi) = K(\phi)(vQ' - Q) \propto \phi^{-2}. \quad (\text{A7})$$

Generically one expects

$$\tilde{\varepsilon}_\phi(v_0) \equiv v_0 Q'(v_0) - Q(v_0) \neq 0, \quad (\text{A8})$$

and we temporarily make this additional assumption. Then we obtain

$$K(\phi) \propto \phi^{-2} \text{ as } \phi \rightarrow \infty. \quad (\text{A9})$$

This is somewhat more general than the function $K(\phi) = \text{const} \cdot \phi^{-2}$ usually considered in k -essence models.

We may consider Lagrangians $p = K(\phi)Q(v)$ with the function $K(\phi)$ of the form

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0. \quad (\text{A10})$$

Let us now derive a sharp condition for the existence of an asymptotically stable solution $\{v(\phi), R(\phi)\}$ in this case. We use the ansatz

$$v(\phi) = v_0 - A(\phi), \quad R(\phi) = R_0 - B(\phi), \quad (\text{A11})$$

where by assumption the unknown functions $A(\phi), B(\phi)$ tend to zero at $\phi \rightarrow \infty$. After deriving and solving the equations for $A(\phi)$ and $B(\phi)$, we will need to verify this assumption.

Since the left-hand side of Eq. (35) is $-A'$, it tends to zero faster than ϕ^{-1} . On the other hand, assuming that $c_s^2(v_0) \neq 0$, we find that the right-hand side of Eq. (35) contains leading terms of order ϕ^{-1} , such as $(\ln K)_{,\phi}$ and \sqrt{K} . Hence, these terms must cancel, which entails

$$3\kappa \sqrt{\frac{\tilde{\varepsilon}_\phi(v_0)}{1 - R_0}} = \frac{2v_0}{1 + w_\phi(v_0)} = \frac{2v_0}{1 + w_m} = \frac{2\tilde{\varepsilon}_\phi(v_0)}{Q'(v_0)}. \quad (\text{A12})$$

Since v_0 is determined from Eq. (A6), this condition fixes the value of R_0 ,

$$R_0 = 1 - \frac{9\kappa^2}{4} \frac{Q'^2(v_0)}{v_0 Q'(v_0) - Q(v_0)}. \quad (\text{A13})$$

The requirement that the values of R_0 be between 0 and 1 further restricts the possible functions $Q(v)$. Using Eq. (A12), the condition $R_0 > 0$ can be expressed equivalently as

$$Q(v_0) < \frac{4}{9\kappa^2} v_0^2 \frac{w_m}{(1 + w_m)^2}. \quad (\text{A14})$$

No tracker solution is possible if this condition is violated.

The equations for $A(\phi)$ and $B(\phi)$ are now found by linearizing the equations (35)–(36). For brevity, we rewrite these equations as

$$\frac{dv}{d \ln \phi} = -\Lambda_1(v) \frac{d \ln K}{d \ln \phi} - \Lambda_2(v) \sqrt{\frac{1 + K_0(\phi)}{1 - R}}, \quad (\text{A15})$$

$$\frac{dR}{d \ln \phi} = -\Lambda_3(v, R) \sqrt{1 + K_0(\phi)} (w_m - w_\phi(v)), \quad (\text{A16})$$

where the auxiliary functions $\Lambda_1, \Lambda_2, \Lambda_3$ are defined by

$$\Lambda_1(v) \equiv \frac{c_s^2(v)v}{1 + w_\phi(v)} = \frac{\tilde{\varepsilon}_\phi}{vQ''(v)}, \quad (\text{A17})$$

$$\Lambda_2(v) \equiv 3\kappa c_s^2(v) \sqrt{\tilde{\varepsilon}_\phi(v)}, \quad (\text{A18})$$

$$\Lambda_3(v, R) \equiv \frac{3\kappa}{v} R \sqrt{1 - R} \sqrt{\tilde{\varepsilon}_\phi(v)} = \frac{R \sqrt{1 - R}}{v c_s^2(v)} \Lambda_2(v). \quad (\text{A19})$$

Note that Eq. (A12) is equivalent to

$$2\Lambda_1(v_0) = \frac{\Lambda_2(v_0)}{\sqrt{1 - R_0}}. \quad (\text{A20})$$

Substituting the ansatz (A11) into Eqs. (A15)–(A16), using the identity (A20), and keeping only the leading linear terms, we find

$$\frac{dA}{d \ln \phi} = (\phi K'_0 + K_0) \Lambda_1(v_0) - \alpha_0 A - \frac{\Lambda_1(v_0)B}{1 - R_0}, \quad (\text{A21})$$

$$\frac{dB}{d \ln \phi} = \Lambda_3(v_0, R_0) w'_\phi(v_0) A, \quad (\text{A22})$$

where we defined the auxiliary constant α_0 by

$$\alpha_0 \equiv \frac{\Lambda'_2(v_0)}{\sqrt{1 - R_0}} - 2\Lambda'_1(v_0) = \frac{1 - w_m}{1 + w_m}. \quad (\text{A23})$$

For the moment, we assume additionally that

$$w'_\phi(v_0) \equiv \left(\frac{Q}{vQ' - Q} \right)'_{v=v_0} = \left(1 - \frac{w_m}{c_s^2(v_0)} \right) \frac{1 + w_m}{v_0} \neq 0. \quad (\text{A24})$$

Differentiating Eq. (A22) with respect to $\ln \phi$ and substituting into Eq. (A21), we find a closed second-order equation for $A(\phi)$,

$$\frac{d^2 B}{d(\ln \phi)^2} + \alpha_0 \frac{dB}{d \ln \phi} + \beta_0 B = \gamma_0 \left[\frac{dK_0}{d \ln \phi} + K_0 \right], \quad (\text{A25})$$

where the constant coefficients β_0, γ_0 are defined by

$$\begin{aligned} \gamma_0 &\equiv \Lambda_1(v_0) \Lambda_3(v_0, R_0) w'_\phi(v_0) \\ &= 2 \frac{c_s^2(v_0) - w_m}{1 + w_m} w_m^2 R_0 (1 - R_0), \end{aligned} \quad (\text{A26})$$

$$\beta_0 \equiv \frac{\gamma_0}{1 - R_0} = 2 \frac{c_s^2(v_0) - w_m}{1 + w_m} w_m^2 R_0. \quad (\text{A27})$$

The general solution of Eq. (A25) is the sum of an inhomogeneous solution and the general solution of the homogeneous equation. Homogeneous solutions are stable if both roots $\lambda_{1,2}$ of the characteristic equation

$$\lambda^2 + \alpha_0 \lambda + \beta_0 = 0 \quad (\text{A28})$$

have negative real parts,

$$\text{Re}(\lambda_1) < 0, \quad \text{Re}(\lambda_2) < 0. \quad (\text{A29})$$

This will be the case if

$$\alpha_0 > 0, \quad \beta_0 > 0, \quad (\text{A30})$$

which is equivalent to the conditions

$$|w_m| < 1, \quad c_s^2(v_0) > w_m. \quad (\text{A31})$$

An inhomogeneous solution of Eq. (A25) can be expressed as

$$B(\phi) = B_1(\phi) \phi^{\lambda_1} + B_2(\phi) \phi^{\lambda_2}, \quad (\text{A32})$$

$$B_1(\phi) \equiv \frac{\gamma_0}{\lambda_1 - \lambda_2} \int^\phi \phi^{-\lambda_1 - 1} (\phi K'_0 + K_0) d\phi, \quad (\text{A33})$$

$$B_2(\phi) \equiv \frac{\gamma_0}{\lambda_2 - \lambda_1} \int^\phi \phi^{-\lambda_2 - 1} (\phi K'_0 + K_0) d\phi. \quad (\text{A34})$$

Since the function $K_0(\phi)$ tends to zero at $\phi \rightarrow \infty$ by assumption, the inhomogeneous solution also tends to zero at $\phi \rightarrow \infty$ as long as the condition (A29) holds. This is straightforward to show by assuming an upper bound

$$|\phi K'_0 + K_0| < M \text{ for all } \phi > \phi_M, \quad (\text{A35})$$

where ϕ_M can be chosen for any $M > 0$. Then the inhomogeneous solution $B(\phi)$ is bounded for $\phi > \phi_M$ by

$$|B(\phi)| < \text{const} \cdot M + \text{const} \cdot \phi^{\lambda_1} + \text{const} \cdot \phi^{\lambda_2}, \quad (\text{A36})$$

which means that $B(\phi) \rightarrow 0$ at $\phi \rightarrow \infty$.

Under the same assumptions, the function $A(\phi)$ will have the same behavior at $\phi \rightarrow \infty$. We conclude that asymptotically stable solutions $\{v(\phi), R(\phi)\}$ approaching $\{v_0, R_0\}$ exist under the assumption $c_s^2(v_0) \neq 0$ and the further conditions (A6), (A8), (A12), (A24), and (A31).⁴ These conditions are similar to those derived in Ref. [3] under a more restrictive assumption $K(\phi) = \text{const} \cdot \phi^{-2}$. Let us now investigate whether these assumptions can be relaxed further.

2. Energy density $\varepsilon_m \propto \phi^{-2}$ and $R_0 \neq 1$, marginal cases

The last assumption used in the derivation of the stability condition (A31) was Eq. (A24). If $c_s^2(v_0) = w_m$ while all the other assumptions hold, we have $w'_\phi(v_0) = 0$ and the equation (A22) for $B(\phi)$ is modified. We may then rewrite Eqs. (A21)–(A22) as

$$\frac{dA}{d \ln \phi} = (\phi K'_0 + K_0) \Lambda_1(v_0) - \alpha_0 A - \frac{\Lambda_1(v_0)B}{1 - R_0}, \quad (\text{A37})$$

$$\frac{dB}{d \ln \phi} = O(A^2). \quad (\text{A38})$$

Differentiating the first equation with respect to $\ln \phi$, we obtain

$$\frac{d^2 A}{d(\ln \phi)^2} = \phi (\phi K_0)'' \Lambda_1(v_0) - \alpha_0 \frac{dA}{d \ln \phi} + O(A^2). \quad (\text{A39})$$

The second-order terms $O(A^2)$ can be disregarded for the stability analysis. Since the characteristic equation

$$\lambda^2 + \alpha_0 \lambda = 0 \quad (\text{A40})$$

has a zero root, the general solution $\{A(\phi), B(\phi)\}$ will not tend to zero at $\phi \rightarrow \infty$. Hence, no asymptotically stable solutions exist when the condition (A31) is violated.

Another assumption, $c_s^2(v_0) \neq 0$, was used to derive Eq. (A12) that determines the allowed value of R_0 . Let

⁴ This is case 1 in Sec. V A.

us briefly consider the possibility $c_s^2(v_0) = 0$. (We note that $v \neq v_0$ on actual trajectories, so stability will hold as long as the trajectories $v(\phi)$ do not reach the regime $c_s^2(v) \leq 0$.) If

$$c_s^2(v_0) = \frac{Q'(v_0)}{v_0 Q''(v_0)} = 0, \quad (\text{A41})$$

then $Q'(v_0) = 0$ and the asymptotic equation of state is

$$w_\phi(v_0) = \frac{Q(v_0)}{v_0 Q'(v_0) - Q(v_0)} = -1 \quad (\text{A42})$$

as long as $Q(v_0) \neq 0$. However, we assumed a matter component with $w_m \neq -1$, and so we discard the possibility that $Q(v_0) \neq 0$. If, on the other hand, $Q(v_0) = 0$, then we must also have $\tilde{\varepsilon}_\phi(v_0) = 0$. Thus $c_s^2(v_0) \neq 0$ is justified given that $\tilde{\varepsilon}_\phi(v_0) \neq 0$.

Relaxing the assumption $\tilde{\varepsilon}_\phi(v_0) \neq 0$ requires some more work. If $\tilde{\varepsilon}_\phi(v_0) = 0$, then we cannot conclude that $K(\phi) \propto \phi^{-2}$ at $\phi \rightarrow \infty$; the function $K(\phi)$ remains undetermined even though we know that $\varepsilon_\phi(\phi) = K(\phi)\tilde{\varepsilon}_\phi(v) \propto \phi^{-2}$. The analysis after Eq. (A6) needs to be modified as follows. The finiteness of w_ϕ ,

$$w_\phi(v_0) = \lim_{v \rightarrow v_0} \frac{Q(v)}{\tilde{\varepsilon}_\phi(v)} < \infty, \quad (\text{A43})$$

requires that $Q(v_0) = 0$ and thus (since $v_0 \neq 0$) also $Q'(v_0) = 0$. In general, we may suppose that $Q(v)$ has an expansion

$$Q(v) = \frac{Q_0}{nv_0} (v - v_0)^n [1 + O(v - v_0)], \quad (\text{A44})$$

where Q_0 is a nonzero constant and $n \geq 2$. In this case we have the expansions

$$\tilde{\varepsilon}_\phi(v) = Q_0 (v - v_0)^{n-1} [1 + O(v - v_0)], \quad (\text{A45})$$

$$w_\phi(v) = \frac{v - v_0}{nv_0} [1 + O(v - v_0)], \quad (\text{A46})$$

$$c_s^2(v) = \frac{v - v_0}{(n-1)v_0} [1 + O(v - v_0)]. \quad (\text{A47})$$

It follows that $w_\phi(v_0) = 0$, so the only possibility for tracking is $w_m = 0$. Also, the only admissible solutions are those with $v(\phi) > v_0$, meaning that $A(\phi) < 0$ and $Q_0 > 0$. Let us now perform a stability analysis of these solutions. Substituting the ansatz (A11) into Eqs. (35)–(36) and keeping only the leading terms in the perturbation variables $A(\phi)$ and $B(\phi)$, we obtain

$$\frac{dA}{d\phi} = \frac{(-A)}{n-1} \frac{K'}{K} + \frac{3\kappa}{(n-1)v_0} \sqrt{\frac{K(\phi)Q_0}{1-R_0}} (-A)^{(n+1)/2}, \quad (\text{A48})$$

$$\frac{dB}{d\phi} = -\frac{3\kappa}{nv_0^2} R_0 \sqrt{1-R_0} \sqrt{K(\phi)Q_0} (-A)^{(n+1)/2}. \quad (\text{A49})$$

For the purposes of a stability analysis, it is sufficient to note that Eq. (A48) does not involve $B(\phi)$. One can solve

Eq. (A48) explicitly for $A(\phi)$ and find such $K(\phi)$ that the general solution for $A(\phi)$ tends to zero at $\phi \rightarrow \infty$; for instance, $K(\phi) \propto \phi^r$ with $r > -2$. However, the general solution for $B(\phi)$ is

$$B(\phi) = B_0 - \text{const} \cdot \int_{\phi_0}^{\phi} \sqrt{K(\phi)} (-A)^{(n+1)/2} d\phi, \quad (\text{A50})$$

where B_0 is an arbitrary integration constant. It follows that $B(\phi)$ will either diverge or tend to an arbitrary constant of integration at $\phi \rightarrow \infty$. Hence, the general perturbation will not tend to zero at large ϕ . We conclude that no asymptotically stable solutions exist when $\tilde{\varepsilon}_\phi(v_0) = 0$.

3. Energy density $\varepsilon_m \propto \phi^{-2}$ and $R_0 = 1$, main case

We use the ansatz $R(\phi) = 1 - B(\phi)$, where the function $B(\phi)$ is positive and tends to zero monotonically as $\phi \rightarrow \infty$. Since $dR/d\phi > 0$, it follows from Eq. (31) that $w_m < w_\phi(v(\phi))$ for all sufficiently large ϕ . Thus, any asymptotically stable solutions will necessarily satisfy the condition

$$w_m \leq w_\phi(v_0). \quad (\text{A51})$$

Since $R \rightarrow 1$ as $\phi \rightarrow \infty$, we have $\varepsilon_{\text{tot}}(\phi) \propto \varepsilon_m(\phi) \propto \phi^{-2}$, so we may write

$$\varepsilon_{\text{tot}}(\phi) \approx E_0 \phi^{-2}, \quad \phi \rightarrow \infty, \quad (\text{A52})$$

where E_0 is a nonzero constant. The value of E_0 can be related to other parameters by using Eq. (25), rewritten as

$$\frac{d \ln \varepsilon_m}{d \ln \phi} = -\frac{3\kappa}{v} \sqrt{\phi^2 \varepsilon_m} (1 + w_m), \quad (\text{A53})$$

which yields, in the limit $\phi \rightarrow \infty$,

$$2 = \frac{3\kappa}{v_0} \sqrt{E_0} (1 + w_m). \quad (\text{A54})$$

Expressing ε_{tot} through ε_ϕ , we have

$$E_0 \phi^{-2} \approx \varepsilon_{\text{tot}} = \frac{\varepsilon_\phi}{1-R} = \frac{\tilde{\varepsilon}_\phi(v) K(\phi)}{B}; \quad (\text{A55})$$

hence

$$B(\phi) \approx \frac{\tilde{\varepsilon}_\phi(v) \phi^2 K(\phi)}{E_0}, \quad \phi \rightarrow \infty. \quad (\text{A56})$$

We now assume that $\tilde{\varepsilon}_\phi(v_0) \neq 0$; the case $\tilde{\varepsilon}_\phi(v_0) = 0$ will be considered later. If $\tilde{\varepsilon}_\phi(v_0) \neq 0$, it follows that

$$B(\phi) \approx \frac{\tilde{\varepsilon}_\phi(v_0)}{E_0} \phi^2 K(\phi), \quad \phi \rightarrow \infty. \quad (\text{A57})$$

Rewriting Eq. (31) as

$$\frac{d \ln (1-R)}{d \ln \phi} = \frac{3\kappa R}{v} \sqrt{\phi^2 \varepsilon_{\text{tot}}} (w_m - w_\phi(v)) \quad (\text{A58})$$

and substituting Eqs. (A52) and (A57), we find for large ϕ

$$\frac{d \ln (\phi^2 K(\phi))}{d \ln \phi} \approx \frac{3\kappa \sqrt{E_0}}{v_0} (w_m - w_\phi(v)) = 2 \frac{w_m - w_\phi(v)}{1 + w_m}. \quad (\text{A59})$$

It is now clear that the possible asymptotic behavior of $K(\phi)$ at $\phi \rightarrow \infty$ depends on whether $w_\phi(v)$ tends to w_m at large ϕ , i.e. on whether or not $w_\phi(v_0) = w_m$. (We note that the value of v_0 is yet to be determined by the analysis that follows.)

Considering the interesting case $w_\phi(v_0) \neq w_m$, we find that the right-hand side of Eq. (A59) tends to a negative constant as $\phi \rightarrow \infty$. Denoting that constant by $-\mu$, where

$$\mu \equiv 2 \frac{w_\phi(v_0) - w_m}{1 + w_m} > 0, \quad (\text{A60})$$

and integrating Eq. (A59), we infer the following asymptotic behavior of $K(\phi)$,

$$K(\phi) \propto \phi^{-2-\mu} K_0(\phi), \quad \phi \rightarrow \infty, \quad (\text{A61})$$

where $K_0(\phi)$ is an auxiliary function that satisfies

$$\lim_{\phi \rightarrow \infty} \frac{d \ln K_0(\phi)}{d \ln \phi} = 0. \quad (\text{A62})$$

This condition is equivalent to

$$\lim_{\phi \rightarrow \infty} \frac{\ln K_0(\phi)}{\ln \phi} = 0. \quad (\text{A63})$$

Thus, the function $K_0(\phi)$ may go to a constant at large ϕ , or may grow or decay slower than any power of ϕ ; examples of admissible functions $K_0(\phi)$ are

$$K_0(\phi) = (\ln \phi)^p; \quad K_0(\phi) = \exp(C_1 (\ln \phi)^s), \quad |s| < 1. \quad (\text{A64})$$

With any such $K_0(\phi)$, solutions of the currently considered type are possible only for Lagrangians $p = K(\phi)Q(v)$ with

$$K(\phi) = \phi^{-2\alpha} K_0(\phi), \quad (\text{A65})$$

where

$$\alpha \equiv \frac{2 + \mu}{2} = \frac{1 + w_\phi(v_0)}{1 + w_m} > 1. \quad (\text{A66})$$

For a given Lagrangian of this type, the possible values of v_0 are fixed by Eq. (A66). If Eq. (A66) is not satisfied for any such v_0 , solutions of this type do not exist. The value $w_\phi(v_0)$ is determined by Eq. (A66) as

$$w_\phi(v_0) = (1 + w_m) \alpha - 1. \quad (\text{A67})$$

Since $w_m \neq -1$, we must have $w_\phi(v_0) \neq -1$ also.

It remains to investigate the asymptotic stability of the general solution. Since $B(\phi)$ must satisfy Eq. (A56), we may write an ansatz

$$B(\phi) = \frac{\tilde{\varepsilon}_\phi(v_0)}{E_0} \phi^2 K(\phi) (1 + C(\phi)), \quad (\text{A68})$$

where $C(\phi)$ is a new perturbation variable. Hence, we substitute Eq. (A65) together with the ansatz

$$v(\phi) = v_0 - A(\phi), \quad (\text{A69})$$

$$R(\phi) = 1 - \frac{\tilde{\varepsilon}_\phi(v_0)}{E_0} \phi^2 K(\phi) (1 + C(\phi)), \quad (\text{A70})$$

$$\varepsilon_{\text{tot}}(v, \phi) = E_0 \phi^{-2} \frac{\tilde{\varepsilon}_\phi(v)}{\tilde{\varepsilon}_\phi(v_0)} \frac{1}{1 + C(\phi)}, \quad (\text{A71})$$

into Eqs. (35) and (A58). Using Eqs. (A54), (A60), and (A65), we obtain at an intermediate step the equations

$$\begin{aligned} \frac{dA}{d\phi} = & \left[-\frac{2\alpha}{\phi} + (\ln K_0)' \right] \Lambda_1(v) \\ & + \frac{\phi^{-1}}{\sqrt{1+C}} \frac{2v_0}{1+w_m} \frac{\Lambda_2(v)}{3\kappa \sqrt{\tilde{\varepsilon}_\phi(v_0)}}, \end{aligned} \quad (\text{A72})$$

$$\begin{aligned} \frac{1}{1+C} \frac{dC}{d\phi} = & \mu \phi^{-1} - (\ln K_0)' \\ & - \frac{\mu \phi^{-1}}{\sqrt{1+C}} \frac{\Lambda_4(v)}{\Lambda_4(v_0)} [1 + O(\phi^{-\mu})], \end{aligned} \quad (\text{A73})$$

where the functions $\Lambda_1(v)$ and $\Lambda_2(v)$ were defined by Eqs. (A17)–(A18), while the new auxiliary function $\Lambda_4(v)$ is defined by

$$\Lambda_4(v) \equiv 3\kappa \sqrt{\tilde{\varepsilon}_\phi(v)} \frac{w_\phi(v) - w_m}{v}. \quad (\text{A74})$$

In the present case, the identity

$$2\alpha \Lambda_1(v_0) = \frac{2v_0}{1+w_m} \frac{\Lambda_2(v_0)}{3\kappa \sqrt{\tilde{\varepsilon}_\phi(v_0)}} \quad (\text{A75})$$

holds due to Eq. (A66).

We now linearize Eqs. (A72)–(A73) with respect to the perturbation variables A and C . To simplify the linearized equations, we use Eqs. (39), (A54), and the definition (A66) of α . (We note that $\Lambda_1(v_0) \neq 0$; otherwise, we would have $c_s^2(v_0) = 0$, which contradicts the earlier assumptions $\tilde{\varepsilon}_\phi(v_0) \neq 0$ and $w_\phi(v_0) \neq -1$.) After some algebra, we find (to the leading order)

$$\begin{aligned} \frac{dA}{d \ln \phi} = & 2\alpha \Lambda_2(v_0) \left(\frac{\Lambda_1(v)}{\Lambda_2(v)} \right)'_{v_0} A \\ & + \Lambda_1(v_0) \left[\frac{d \ln K_0}{d \ln \phi} - \alpha C \right], \end{aligned} \quad (\text{A76})$$

$$\frac{dC}{d \ln \phi} = -\frac{d \ln K_0}{d \ln \phi} + \frac{1}{2} \mu C + \mu \frac{\Lambda_4'(v_0)}{\Lambda_4(v_0)} A + \mu \frac{\tilde{\varepsilon}_\phi(v_0)}{E_0} \phi^{-\mu}. \quad (\text{A77})$$

This is an inhomogeneous linear system for $A(\phi)$ and $C(\phi)$. The analysis of the asymptotic stability is similar to that after Eq. (A29). Since all the inhomogeneous terms are decaying at $\phi \rightarrow \infty$, it suffices to require that both the eigenvalues of the homogeneous system have

negative real parts. For a homogeneous system of the form

$$\frac{dA}{d \ln \phi} = \beta_1 A + \beta_2 C, \quad (\text{A78})$$

$$\frac{dC}{d \ln \phi} = \gamma_1 A + \gamma_2 C, \quad (\text{A79})$$

the characteristic equation is

$$\lambda^2 - (\beta_1 + \gamma_2) \lambda + (\beta_1 \gamma_2 - \beta_2 \gamma_1) = 0, \quad (\text{A80})$$

and the stability conditions are

$$\beta_1 + \gamma_2 < 0, \quad \beta_1 \gamma_2 - \beta_2 \gamma_1 > 0. \quad (\text{A81})$$

Presently, the constants $\beta_1, \beta_2, \gamma_1, \gamma_2$ can be read off from Eqs. (A76)–(A77); simplifying, we obtain

$$\beta_1 = -\alpha \frac{1 - w_\phi(v_0)}{1 + w_\phi(v_0)}, \quad \beta_2 = -\alpha \frac{c_s^2(v_0) v_0}{1 + w_\phi(v_0)}, \quad (\text{A82})$$

$$\gamma_1 = \frac{\mu}{v_0 c_s^2(v_0)} \left[\frac{(c_s^2(v_0) - w_\phi(v_0)) (1 + w_m)}{w_\phi(v_0) - w_m} + \frac{1 - w_\phi(v_0)}{2} \right], \quad \gamma_2 = \frac{\mu}{2}. \quad (\text{A83})$$

The stability conditions (A81) can be simplified to

$$\frac{1 - w_\phi(v_0)}{1 + w_\phi(v_0)} > \frac{\mu}{2\alpha}, \quad \frac{c_s^2(v_0) - w_\phi(v_0)}{w_\phi(v_0) - w_m} > 0. \quad (\text{A84})$$

Since $w_\phi(v_0) > w_m$ for solutions of the present type, while $2\alpha = \mu + 2$, the stability conditions (together with the condition $w_\phi(v_0) > w_m$) are

$$w_m < w_\phi(v_0) < \frac{1}{1 + \mu}, \quad c_s^2(v_0) > w_\phi(v_0). \quad (\text{A85})$$

Using Eq. (A67), we can transform the first of these conditions into a condition for α :

$$1 < \alpha < \frac{1}{2} + \frac{1}{1 + w_m}, \quad c_s^2(v_0) > w_\phi(v_0). \quad (\text{A86})$$

The first inequality above will define a nonempty interval of α only if $|w_m| < 1$. These are the final conditions for the asymptotic stability of the solutions obtained under the assumptions $\tilde{\varepsilon}_\phi(v_0) \neq 0$, $w_\phi(v_0) \neq w_m$, and (A66).⁵

4. Energy density $\varepsilon_m \propto \phi^{-2}$ and $R_0 = 1$, marginal cases

The analysis in the previous section used the assumptions $\tilde{\varepsilon}_\phi(v_0) \neq 0$ and $w_\phi(v_0) \neq w_m$. In this section we lift these assumption, in the reverse order used.

If $w_\phi(v_0) = w_m$ while $\tilde{\varepsilon}_\phi(v_0) \neq 0$, then we may continue the arguments starting with Eq. (A59). Note that Eqs. (A54) and (A57) still hold. Since the right-hand side of Eq. (A59) tends to zero at $\phi \rightarrow \infty$, it follows that

$$\lim_{\phi \rightarrow \infty} \frac{d \ln (\phi^2 K(\phi))}{d \ln \phi} = 0. \quad (\text{A87})$$

This condition is equivalent to

$$\lim_{\phi \rightarrow \infty} \frac{\ln K(\phi)}{\ln \phi} = -2. \quad (\text{A88})$$

Also, according to Eq. (A57) we can have $B(\phi) \rightarrow 0$ only if

$$\lim_{\phi \rightarrow \infty} \phi^2 K(\phi) = 0. \quad (\text{A89})$$

So the function $K(\phi)$ cannot have a power-law asymptotic other than ϕ^{-2} ; more precisely, for any $\varepsilon > 0$ and for large enough ϕ we must have

$$K(\phi) < \phi^{-2+\varepsilon}, \quad K(\phi) > \phi^{-2-\varepsilon}, \quad \phi \rightarrow \infty. \quad (\text{A90})$$

However, a non-power law asymptotic behavior at $\phi \rightarrow \infty$ is still admissible, for instance $K(\phi) \propto \phi^{-2} (\ln \phi)^{-s}$, where $s > 0$ to allow $B(\phi) \rightarrow 0$ according to Eq. (A57). Rather than assume a particular form of $K(\phi)$, we will perform the analysis for arbitrary $K(\phi)$ satisfying Eq. (A88).

We again use the ansatz (A68) to linearize Eqs. (35) and (A58). After some algebra, we find (to the leading order)

$$\frac{dA}{d \ln \phi} = \left[-2 + \frac{d \ln (\phi^2 K)}{d \ln \phi} \right] \Lambda_1(v) + \frac{\Lambda_2(v)}{\sqrt{1+C}} \sqrt{\frac{E_0}{\tilde{\varepsilon}_\phi(v_0)}}, \quad (\text{A91})$$

$$\frac{dC}{d \ln \phi} = -\frac{d \ln (\phi^2 K)}{d \ln \phi} - \frac{\Lambda_4(v)}{\sqrt{1+C}} \sqrt{\frac{E_0}{\tilde{\varepsilon}_\phi(v_0)}}, \quad (\text{A92})$$

where the auxiliary functions $\Lambda_1(v)$, $\Lambda_2(v)$, and $\Lambda_4(v)$ were defined above by Eqs. (A17), (A18), and (A74). Since $w_\phi(v_0) = w_m$ and $c_s^2(v_0) \neq 0$, we have the relationship,

$$2\Lambda_1(v_0) = \Lambda_2(v_0) \sqrt{\frac{E_0}{\tilde{\varepsilon}_\phi(v_0)}} \neq 0, \quad (\text{A93})$$

and then, assuming for the moment that $\Lambda'_4(v_0) \neq 0$, we can linearize Eqs. (A91)–(A92) as

$$\frac{dA}{d \ln \phi} = 2\Lambda_2(v_0) \left(\frac{\Lambda_1(v)}{\Lambda_2(v)} \right)'_{v_0} A - C \Lambda_1(v_0) + \frac{d \ln (\phi^2 K)}{d \ln \phi} \Lambda_1(v_0), \quad (\text{A94})$$

$$\frac{dC}{d \ln \phi} = \frac{2v_0}{1+w_m} \frac{\Lambda'_4(v_0)}{3\kappa \sqrt{\tilde{\varepsilon}_\phi(v_0)}} A - \frac{d \ln (\phi^2 K)}{d \ln \phi}. \quad (\text{A95})$$

⁵ This is case 3 in Sec. V A.

The stability analysis proceeds as before, since all the inhomogeneous terms are decaying at $\phi \rightarrow \infty$. The resulting conditions are simplified to

$$\frac{1 - w_m}{1 + w_m} > 0, \quad \frac{c_s^2 - w_m}{1 + w_m} > 0, \quad (\text{A96})$$

and further to

$$|w_m| < 1, \quad c_s^2 > w_m. \quad (\text{A97})$$

These conditions are the same as the standard stability conditions for a tracker solution. Under these conditions, a tracker solution with $w_\phi(v_0) = w_m$ exists as long as $K(\phi)$ satisfies Eqs. (A88)–(A89).⁶

Finally, we analyze the case $\tilde{\varepsilon}_\phi(v_0) = 0$. Since Eqs. (A52), (A54), and (A56) still hold for an asymptotically stable solution, we are motivated to use the ansatz

$$v(\phi) = v_0 - A(\phi), \quad (\text{A98})$$

$$R(\phi) = 1 - B(v, \phi), \quad (\text{A99})$$

$$B(v, \phi) \equiv \frac{\tilde{\varepsilon}_\phi(v)}{E_0} \phi^2 K(\phi) (1 + C(\phi)). \quad (\text{A100})$$

We first derive the *exact* equations of motion for the variables $A(\phi), C(\phi)$ from Eqs. (35) and (A58):

$$\frac{dA}{d \ln \phi} = \frac{v c_s^2(v)}{1 + w_\phi(v)} \frac{d \ln K}{d \ln \phi} + \frac{2v_0}{1 + w_m} \frac{c_s^2(v)}{\sqrt{1 + C}}, \quad (\text{A101})$$

$$\begin{aligned} \frac{d \ln(1 + C)}{d \ln \phi} &= \frac{2}{v} A + \frac{2v_0}{v} \left(\frac{1}{\sqrt{1 + C}} - 1 \right) \\ &\quad - \frac{2v_0}{v} \frac{B(v, \phi)}{\sqrt{1 + C(\phi)}} \frac{w_m - w_\phi(v)}{1 + w_m}. \end{aligned} \quad (\text{A102})$$

Then the stability analysis consists of checking that the general solution involves functions $A(\phi), B(v, \phi), C(\phi)$ that decay as $\phi \rightarrow \infty$. Since $v_0 \neq 0$, the expansions (A45)–(A47) hold with $n \geq 2$; we note that $A < 0$ to guarantee $c_s^2 > 0$, and that $w_\phi(v_0) = 0$. The leading-order terms in Eq. (A101) are

$$\frac{d|A|}{d \ln \phi} = -\frac{|A|}{n-1} \left(\frac{d \ln K(\phi)}{d \ln \phi} + \frac{2}{1 + w_m} \right), \quad (\text{A103})$$

and the general solution is

$$|A| = A_0 \left[\phi^{\frac{2}{1+w_m}} K(\phi) \right]^{-\frac{1}{n-1}}. \quad (\text{A104})$$

Since $n \geq 2$, solutions $A(\phi)$ decay at $\phi \rightarrow \infty$ as long as

$$K(\phi) \phi^{\frac{2}{1+w_m}} \rightarrow \infty, \quad \phi \rightarrow \infty. \quad (\text{A105})$$

The function $B(v, \phi)$ is then expressed as

$$B(v, \phi) = \frac{Q_0 A_0^{n-1}}{E_0} \phi^{\frac{2w_m}{1+w_m}} (1 + C(\phi)), \quad (\text{A106})$$

and its decay at $\phi \rightarrow \infty$ requires that $-1 < w_m < 0$. The leading terms of Eq. (A102) are

$$\frac{dC}{d \ln \phi} = \frac{2}{v_0} A - C - \left(1 - \frac{1}{2} C \right) B(v, \phi) \frac{w_m}{1 + w_m}.$$

Since the homogeneous solution $C(\phi) \propto \phi^{-1}$ decays as a power of ϕ , while the inhomogeneous terms all decay at $\phi \rightarrow \infty$, the general solution $C(\phi)$ will also decay at $\phi \rightarrow \infty$. Thus, we find a family of asymptotically stable solutions corresponding to a value v_0 such that Eq. (A44) holds, in case $w_m < 0$ and for Lagrangians with $K(\phi)$ that either does not decay at large ϕ , or decays slower than $\phi^{-2/(1+w_m)}$.⁷

5. Domination by k -essence, $v_0 \neq 0$, main case

We now consider the case $R_0 = 0$. In this case, the matter component becomes subdominant at late times, so $\varepsilon_{\text{tot}} \approx \varepsilon_\phi$ at $\phi \rightarrow \infty$. According to Eq. (26), we have at late times

$$\begin{aligned} \frac{d}{d\phi} \varepsilon_\phi(\phi) &= -\frac{3\kappa}{v} \sqrt{\varepsilon_{\text{tot}}} (1 + w_\phi) \varepsilon_\phi \\ &\approx -\frac{3\kappa}{v_0} \varepsilon_\phi^{3/2} (1 + w_\phi(v)), \end{aligned} \quad (\text{A107})$$

thus the asymptotic behavior of $\varepsilon_\phi(\phi)$ depends on whether or not $w_\phi(v_0) = -1$. With $v_0 \neq 0$, one can have $w_\phi(v_0) = -1$ only if $Q'(v_0) = 0$, which entails

$$c_s^2(v_0) = \frac{1}{v_0} \lim_{v \rightarrow v_0} \frac{Q'(v)}{Q''(v)} = \frac{1}{v_0} \lim_{v \rightarrow v_0} \frac{1}{(\ln Q'(v))'} = 0. \quad (\text{A108})$$

Let us postpone the consideration of the case $c_s(v_0) = 0$; thus, presently we have $w_\phi(v_0) \neq -1$. In that case, the asymptotic behavior of $\varepsilon_\phi(\phi)$ and $\varepsilon_{\text{tot}}(\phi)$ can be expressed as

$$\varepsilon_{\text{tot}}(\phi) \approx \varepsilon_\phi(\phi) \approx E_0 \phi^{-2}, \quad (\text{A109})$$

where the constant E_0 is given by

$$3\kappa \sqrt{E_0} = \frac{2v_0}{1 + w_\phi(v_0)}, \quad (\text{A110})$$

due to Eq. (A107). We use Eqs. (35)–(36) to describe asymptotically stable solutions. Since on such solutions $R(\phi)$ approaches zero while remaining positive, we must

⁶ This is case 4 in Sec. V A.

⁷ This is case 5 in Sec. V A.

have $w_\phi(v) < w_m$ at late times. Computing the limit of Eq. (35) as $\phi \rightarrow \infty$ and using Eq. (A110), we find

$$\begin{aligned} 0 &= \lim_{\phi \rightarrow \infty} \frac{dv}{d \ln \phi} = \lim_{\phi \rightarrow \infty} \phi c_s^2(v) \left[\frac{(\ln K)_{,\phi} v}{1 + w_\phi(v)} + 3\kappa \sqrt{\varepsilon_{\text{tot}}} \right] \\ &= \frac{v_0}{1 + w_\phi(v_0)} \lim_{\phi \rightarrow \infty} c_s^2(v) \left[\frac{d \ln K(\phi)}{d \ln \phi} + 2 \right]. \end{aligned} \quad (\text{A111})$$

The right-hand side of Eq. (A111) can vanish at $\phi \rightarrow \infty$ if, for instance, $c_s^2(v_0) = 0$. We postpone the consideration of the case $c_s^2(v_0) = 0$ and presently assume that $c_s^2(v_0) \neq 0$, which (together with $v_0 \neq 0$) also implies $\tilde{\varepsilon}_\phi(v_0) \neq 0$. Then $\varepsilon_\phi(\phi) \propto \phi^{-2}$ entails $K(\phi) \propto \phi^{-2}$ at $\phi \rightarrow \infty$; accordingly, the right-hand side of Eq. (A111) vanishes at $\phi \rightarrow \infty$ due to

$$\lim_{\phi \rightarrow \infty} \frac{d \ln K}{d \ln \phi} = -2. \quad (\text{A112})$$

By absorbing a constant into $Q(v)$ if necessary, we may express $K(\phi)$ as

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0. \quad (\text{A113})$$

This is the familiar form of the function $K(\phi)$, shown by Eq. (A10) in Sec. A 1.

For these $K(\phi)$, the condition (A110) becomes

$$3\kappa \sqrt{E_0} = 3\kappa \sqrt{\tilde{\varepsilon}_\phi(v_0)} = \frac{2v_0}{1 + w_\phi(v_0)}, \quad (\text{A114})$$

which is an equation for determining the admissible values of v_0 . For these v_0 , we linearize Eqs. (35)–(36) using the ansatz

$$v = v_0 - A(\phi), \quad R = B(\phi), \quad (\text{A115})$$

where $A(\phi), B(\phi)$ tend to zero as $\phi \rightarrow \infty$. The manipulations with Eq. (35) are the same as those in Sec. A 1; the result of the linearization is quite similar to Eq. (A21) with $R_0 = 0$ and without the relationship $w_\phi(v_0) = w_m$,

$$\frac{dA}{d \ln \phi} = (\phi K'_0 + K_0) \Lambda_1(v_0) - \frac{1 - w_\phi(v_0)}{1 + w_\phi(v_0)} A - \Lambda_1(v_0) B. \quad (\text{A116})$$

The linearized form of Eq. (36) is

$$\begin{aligned} \frac{dB}{d \ln \phi} &= -\frac{3\kappa \sqrt{E_0}}{v_0} (w_m - w_\phi(v_0)) B \\ &= -2 \frac{w_m - w_\phi(v_0)}{1 + w_\phi(v_0)} B. \end{aligned} \quad (\text{A117})$$

Since the equation for $B(\phi)$ does not involve $A(\phi)$, and since $w_m > w_\phi(v_0)$, all solutions $B(\phi)$ decay, and thus all solutions $A(\phi)$ also decay as long as

$$\frac{1 - w_\phi(v_0)}{1 + w_\phi(v_0)} > 0, \quad (\text{A118})$$

which is equivalent to $|w_\phi(v_0)| < 1$. Therefore, solutions are asymptotically stable under the conditions (A114), $|w_\phi(v_0)| < 1$, $w_m > w_\phi(v_0)$, and $c_s(v_0) \neq 0$.⁸

6. Domination by k -essence, $v_0 \neq 0$, marginal cases

In this section we continue considering the case $R_0 = 0$, $v_0 \neq 0$, and examine the possibility that $c_s^2(v_0) = 0$. In that case, we have $Q'(v_0) = 0$ as well, which fixes admissible values of v_0 . There are two further possibilities: either $Q(v_0) \neq 0$ or $Q(v_0) = 0$.

If $Q(v_0) \equiv Q_0 \neq 0$, then $Q(v)$ can be expanded about $v = v_0$ as

$$Q(v) = Q_0 + Q_1 (v - v_0)^n [1 + O(v - v_0)], \quad (\text{A119})$$

where $n \geq 2$. One readily obtains the expansions

$$\tilde{\varepsilon}_\phi(v) = -Q_0 + nv_0 Q_1 (v - v_0)^{n-1} [1 + O(v - v_0)], \quad (\text{A120})$$

$$w_\phi(v) = -1 + \frac{nv_0 Q_1}{-Q_0} (v - v_0)^{n-1} [1 + O(v - v_0)], \quad (\text{A121})$$

$$c_s^2(v) = \frac{1}{v_0} \frac{v - v_0}{n - 1} [1 + O(v - v_0)]. \quad (\text{A122})$$

It is clear that one must have $K(\phi)Q_0 < 0$ due to the positivity of the energy density. For convenience, let us assume that $K(\phi) > 0$ and $Q_0 < 0$. Substituting the expansions above into Eqs. (35)–(36) together with the ansatz (A115) and neglecting the subleading terms, we obtain

$$\frac{dA}{d\phi} = \frac{1}{v_0} \frac{-A}{n-1} \left[\frac{|Q_0| (\ln K)_{,\phi}}{n Q_1 (-A)^{n-1}} + 3\kappa \sqrt{|Q_0| K(\phi)} \right], \quad (\text{A123})$$

$$\frac{dB}{d\phi} = -\frac{3\kappa}{v_0} B \sqrt{|Q_0| K(\phi)} (w_m + 1). \quad (\text{A124})$$

Since these equations are uncoupled in the leading order, the stability analysis is performed for each equation separately. Integrating Eq. (A124), we find the general solution

$$B(\phi) = \exp \left[C_0 - \frac{3\kappa \sqrt{|Q_0|}}{v_0} (w_m + 1) \int^\phi \sqrt{K(\phi)} d\phi \right], \quad (\text{A125})$$

where C_0 is an integration constant. The general solution $B(\phi)$ will tend to zero if and only if $\int \sqrt{K(\phi)} d\phi$ diverges as $\phi \rightarrow \infty$ and $w_m > -1$. Let us temporarily denote

$$\chi(\phi) \equiv \int^\phi \sqrt{K(\phi)} d\phi, \quad \chi \rightarrow \infty \text{ as } \phi \rightarrow \infty. \quad (\text{A126})$$

⁸ This is case 2 in Sec. V A.

Then we rewrite the first equation as

$$\frac{d}{d\chi} (-A)^{n-1} = -\frac{|Q_0|}{nv_0Q_1} \frac{K'(\phi)}{K^{3/2}} - (-A)^{n-1} \frac{3\kappa}{v_0} \sqrt{|Q_0|}. \quad (\text{A127})$$

(Note that we must have $A < 0$ on solutions, due to the requirement of positivity of c_s^2 .) The general solution $A(\chi)$ can now be written explicitly, but it suffices to observe that $A(\chi)$ will approach zero as $\chi \rightarrow \infty$ if and only if

$$\lim_{\phi \rightarrow \infty} \frac{K'(\phi)}{K^{3/2}} = -2 \lim_{\phi \rightarrow \infty} \frac{d}{d\phi} K^{-1/2} = 0. \quad (\text{A128})$$

This condition is equivalent to

$$\lim_{\phi \rightarrow \infty} \phi \sqrt{K(\phi)} = \infty. \quad (\text{A129})$$

Note that the condition (A126) follows from that of Eq. (A129). To verify this more formally, consider a function $K(\phi)$ such that $\int^\infty \sqrt{K(\phi)} d\phi < \infty$. Then $K^{1/2}(\phi)$ necessarily decays faster than ϕ^{-1} at $\phi \rightarrow \infty$, and so $K^{-1/2}$ grows faster than ϕ at $\phi \rightarrow \infty$. Such $K(\phi)$ cannot satisfy Eq. (A129). Therefore it is sufficient to impose only the condition (A129). This condition is satisfied, for instance, by functions $K(\phi) \propto \phi^s$ with $s > -2$. Thus, we conclude that the solution with $R_0 = 0$ is asymptotically stable under the condition (A129) and assumptions $Q(v_0) \neq 0$, $Q'(v_0) = 0$.⁹

It remains to consider the case $R_0 = 0$, $Q(v_0) = Q'(v_0) = 0$. In that case, similarly to that discussed in Sec. A 2, we may use the expansions (A44)–(A47). It follows that $w_\phi(v_0) = 0$. With the ansatz $v(\phi) = v_0 - A(\phi)$, we find that $A(\phi) < 0$ on physically reasonable solutions. Then the leading terms of Eq. (35) are

$$\frac{d(-A)}{d\phi} = -\frac{(-A)}{n-1} \frac{K'}{K} - \frac{3\kappa\sqrt{Q_0}}{(n-1)v_0} \sqrt{K(\phi)} (-A)^{(n+1)/2}. \quad (\text{A130})$$

Since this equation is independent of B , it suffices to ensure that $A(\phi) \rightarrow 0$ as $\phi \rightarrow \infty$ and subsequently consider the general solution for $R(\phi)$. The general solution for $A(\phi)$ can be easily found by rewriting Eq. (A130) as

$$\frac{d}{d\phi} \left[(-A)^{-(n-1)/2} K^{-1/2} \right] = \frac{3\kappa\sqrt{Q_0}}{2v_0}. \quad (\text{A131})$$

We find

$$(-A)^{(n-1)/2} = \frac{2v_0}{3\kappa\sqrt{Q_0}} \frac{1}{\phi - \phi_0} \frac{1}{\sqrt{K(\phi)}}, \quad (\text{A132})$$

where ϕ_0 is a constant of integration. It follows that $A(\phi) \rightarrow 0$ as $\phi \rightarrow \infty$ if $K(\phi)$ is such that $\phi^2 K(\phi) \rightarrow \infty$. Under this assumption, we find that

$$\varepsilon_\phi = K(\phi) \tilde{\varepsilon}_\phi(v) \propto \phi^{-2}, \quad \phi \rightarrow \infty, \quad (\text{A133})$$

as it should according to Eq. (A109). Now we analyze the general solution for $R(\phi)$. Then the leading terms of Eq. (36) are

$$\frac{dR}{d \ln \phi} = -R \frac{3\kappa\sqrt{E_0}}{v_0} \left(w_m + \frac{A}{nv_0} \right) = -2R \left(w_m + \frac{A}{nv_0} \right), \quad (\text{A134})$$

where we used Eq. (A110). If $w_m = 0$, the right-hand side above is always positive and (since R is always positive) the general solution for $R(\phi)$ cannot approach zero. If $w_m \neq 0$, the general solution for $R(\phi)$ is

$$R(\phi) \propto \phi^{-2w_m} \quad \text{as } \phi \rightarrow \infty. \quad (\text{A135})$$

It follows that the general solution $R(\phi) \rightarrow 0$ at $\phi \rightarrow \infty$ as long as $w_m > 0$. We conclude that an asymptotically stable solution exists in case $Q(v_0) = Q'(v_0) = 0$ if $w_m > 0$ and $\phi^2 K(\phi) \rightarrow \infty$ as $\phi \rightarrow \infty$. The admissible functions $K(\phi)$ are, for instance, $K(\phi) \propto \phi^s$ with $s > -2$.¹⁰

7. Slow motion ($v_0 = 0$), main case ($Q(0) \neq 0$)

Previously we have been assuming that $v_0 \neq 0$. Now we turn to the case $v_0 = 0$, which means that the velocity $\dot{\phi} \equiv v(\phi)$ of the field ϕ approaches zero, albeit sufficiently slowly so that ϕ still reaches arbitrarily large values at late times. We will now obtain the conditions for the existence of asymptotically stable solutions with $v(\phi) \rightarrow 0$ at $\phi \rightarrow \infty$.

The finiteness of the speed of sound at $v \rightarrow 0$,

$$\lim_{v \rightarrow 0} c_s^2(v) = \lim_{v \rightarrow 0} \frac{Q'(v)}{vQ''(v)} < \infty, \quad (\text{A136})$$

requires that $Q'(0) = 0$. Since the important quantity $\tilde{\varepsilon}_\phi(v) = vQ' - Q$ approaches $-Q(0)$ at late times, it is useful to distinguish two possibilities: $Q(0) \neq 0$ and (less generically) $Q(0) = 0$. In this section we consider the generic case, $Q(0) \equiv -Q_0 \neq 0$. Positivity of the energy density requires that $K(\phi)Q_0 > 0$, and we will choose $K(\phi) > 0$ and $Q_0 > 0$.

Under these assumptions, we may expand the function $Q(v)$ near $v = 0$ as

$$Q(v) = -Q_0 + Q_1 v^n [1 + O(v)], \quad (\text{A137})$$

where $n \geq 2$ is the lowest order of the nonvanishing derivative of $Q(v)$ at $v = 0$, and $Q_1 > 0$ because $Q(v)$ is a convex and monotonically growing function of v . Other relevant quantities are then expanded as

$$\tilde{\varepsilon}_\phi(v) = Q_0 + (n-1) Q_1 v^n [1 + O(v)], \quad (\text{A138})$$

$$w_\phi(v) = -1 + \frac{nQ_1}{Q_0} v^n [1 + O(v)], \quad (\text{A139})$$

$$c_s^2(v) = \frac{1}{n-1} [1 + O(v)]. \quad (\text{A140})$$

⁹ This is case 8 in Sec. V A.

¹⁰ This is case 9 in Sec. V A.

It follows that the only possible equation of state is $w_\phi(0) = -1$, indicating a possible de Sitter tracker solution.

The equations of motion (35)–(36) become (neglecting terms of order v)

$$\frac{dv}{d\phi} = -\frac{1}{n-1} \left[\frac{Q_0}{nQ_1 v^{n-1}} \frac{K'}{K} + 3\kappa \sqrt{\frac{K(\phi)Q_0}{1-R}} \right], \quad (\text{A141})$$

$$\frac{dR}{d\phi} = -\frac{3\kappa}{v} R \sqrt{1-R} \sqrt{K(\phi)Q_0} (w_m + 1). \quad (\text{A142})$$

The first step is to investigate the possibility that $R(\phi) \rightarrow 1$ at large ϕ (we will find that this possibility cannot be realized). We note that for $w_m > -1$, the right-hand side of Eq. (A142) always remains negative. Thus, for $w_m > -1$ the general solution $R(\phi)$ cannot tend to 1 at $\phi \rightarrow \infty$, regardless of the behavior of $K(\phi)$ and $v(\phi)$. In case $w_m < -1$, we need to do more work to establish that there are no asymptotically stable solutions with $R_0 = 1$.

Substituting the ansatz $R(\phi) = 1 - B(\phi)$ into Eq. (A142) and assuming that $B \rightarrow 0$, we obtain (omitting terms of order v and B)

$$\frac{d\sqrt{B}}{d\phi} = -\frac{3\kappa}{2v} \sqrt{K(\phi)Q_0} |1 + w_m|. \quad (\text{A143})$$

Changing the variable from ϕ to χ defined by

$$\chi(\phi) \equiv \int^\phi \sqrt{K(\phi)} d\phi, \quad (\text{A144})$$

we find

$$\frac{d\sqrt{B}}{d\chi} = -\frac{3\kappa}{2v} \sqrt{Q_0} |1 + w_m|. \quad (\text{A145})$$

There are now two possibilities: either the integral in Eq. (A144) diverges at $\phi \rightarrow \infty$, or it converges. Accordingly, either $\chi \rightarrow \infty$ or $\chi \rightarrow \chi_0 < \infty$ at $\phi \rightarrow \infty$. In case $\chi \rightarrow \infty$ at $\phi \rightarrow \infty$, we would have

$$\lim_{\chi \rightarrow \infty} \frac{d\sqrt{B}}{d\chi} = 0. \quad (\text{A146})$$

Since the right-hand side in Eq. (A145) tends to infinity at $\phi \rightarrow \infty$, the case $\chi \rightarrow \infty$ is impossible. Thus, the integral in Eq. (A144) must converge at $\phi \rightarrow \infty$. It follows that $K(\phi) \rightarrow 0$ faster than ϕ^{-2} at $\phi \rightarrow \infty$, and then we may express $K(\phi)$ through an auxiliary function $K_0(\phi)$ as

$$K(\phi) = \phi^{-2} K_0(\phi), \quad \lim_{\phi \rightarrow \infty} K_0(\phi) = 0. \quad (\text{A147})$$

Further, we rewrite Eq. (A141) as

$$\frac{dv}{d\ln \phi} = \frac{1}{n-1} \left[\frac{Q_0}{nQ_1 v^{n-1}} \left(2 - \frac{d\ln K_0}{d\ln \phi} \right) - 3\kappa \sqrt{\frac{K_0(\phi)Q_0}{B}} \right]. \quad (\text{A148})$$

By construction,

$$\lim_{\phi \rightarrow \infty} \left(2 - \frac{d\ln K_0}{d\ln \phi} \right) > 2 \quad (\text{A149})$$

(the limit might even be positive infinite if K_0 tends to zero sufficiently quickly). Hence, under the assumptions $v(\phi) \rightarrow 0$ and $B(\phi) \rightarrow 0$ we must have

$$\lim_{\phi \rightarrow \infty} \frac{Q_0}{nQ_1 v^{n-1}} \left(2 - \frac{d\ln K_0}{d\ln \phi} \right) = +\infty. \quad (\text{A150})$$

It then follows by taking the limit $\phi \rightarrow \infty$ of Eq. (A148) that the two terms in the brackets must cancel while both approach infinity. Therefore, at large ϕ we must have the approximate relationship

$$\frac{v^{n-1}(\phi)}{\sqrt{B(\phi)}} \approx \frac{\sqrt{Q_0}}{3\kappa n Q_1} \frac{1}{\sqrt{K_0(\phi)}} \left(2 - \frac{d\ln K_0}{d\ln \phi} \right) \equiv M(\phi). \quad (\text{A151})$$

Due to Eq. (A149), the auxiliary function $M(\phi)$ defined by Eq. (A151) has the properties

$$\lim_{\phi \rightarrow \infty} M(\phi) \sqrt{K_0(\phi)} > \frac{2\sqrt{Q_0}}{3\kappa n Q_1}, \quad \lim_{\phi \rightarrow \infty} M(\phi) = +\infty. \quad (\text{A152})$$

(The first limit may be positive infinite.) Using the function $M(\phi)$, we may rewrite Eq. (A148) as

$$\frac{dv}{d\ln \phi} = \frac{3\kappa \sqrt{K_0(\phi)Q_0}}{n-1} \left[\frac{M}{v^{n-1}} - \frac{1}{\sqrt{B}} \right]. \quad (\text{A153})$$

Expressing \sqrt{B} through v using Eq. (A151) and substituting the resulting expression for \sqrt{B} into Eq. (A143), we find

$$\begin{aligned} \frac{d}{d\ln \phi} \left[\frac{v^{n-1}}{M} \right] &= -\frac{3\kappa}{2v} \sqrt{K_0(\phi)Q_0} |1 + w_m| \\ &= (n-1) \frac{v^{n-2}}{M} \frac{dv}{d\ln \phi} - v^{n-1} M^{-2} \frac{dM}{d\ln \phi}. \end{aligned} \quad (\text{A154})$$

Rewriting the last equation as

$$\frac{3\kappa}{2} M \sqrt{K_0 Q_0} |1 + w_m| = -\frac{n-1}{n} \frac{dv^n}{d\ln \phi} + v^n \frac{d\ln M}{d\ln \phi}, \quad (\text{A155})$$

we note that the left-hand side tends to a positive limit (or to a positive infinity) due to Eq. (A152), while the term $dv^n/d\ln \phi$ tends to zero at $\phi \rightarrow \infty$ and can be neglected. Therefore, for large ϕ we obtain

$$v^n \approx \frac{3\kappa}{2} \sqrt{K_0(\phi)Q_0} |1 + w_m| M^2 \left[\frac{dM}{d\ln \phi} \right]^{-1}. \quad (\text{A156})$$

This relationship is sufficient for our purposes; we will now show that $v(\phi)$ cannot tend to zero at $\phi \rightarrow \infty$. If we assume that $v(\phi) \rightarrow 0$, we must have

$$\lim_{\phi \rightarrow \infty} \frac{\sqrt{K_0(\phi)}}{\frac{d}{d\ln \phi} M^{-1}} = 0. \quad (\text{A157})$$

Using Eq. (A151), we transform this condition into

$$\lim_{\phi \rightarrow \infty} \left[\frac{\frac{1}{2} (\ln K_0)_{,\ln \phi}}{2 - (\ln K_0)_{,\ln \phi}} + \frac{d}{d \ln \phi} \left[2 - \frac{d \ln K_0}{d \ln \phi} \right]^{-1} \right] = \infty. \quad (\text{A158})$$

It is now straightforward to show that the condition (A158) cannot be satisfied by a function $K_0(\phi)$ that tends to zero at $\phi \rightarrow \infty$. Since $(\ln K_0)' \leq 0$ for all ϕ , the function $(\ln K_0)_{,\ln \phi}$ tends to a nonpositive constant or to a negative infinity at $\phi \rightarrow \infty$. Hence, we obtain the bounds

$$-1 < \frac{\frac{1}{2} (\ln K_0)_{,\ln \phi}}{2 - (\ln K_0)_{,\ln \phi}} < 0, \quad 0 < \left[2 - \frac{d \ln K_0}{d \ln \phi} \right]^{-1} < \frac{1}{2}. \quad (\text{A159})$$

The derivative of a bounded function cannot have an infinite limit. Therefore the limit (A158) cannot be infinite. Since the condition (A158) cannot be satisfied, solutions with $v(\phi) \rightarrow 0$ and $B(\phi) \rightarrow 0$ do not exist under the present assumptions.

Having shown that $R_0 = 1$ is impossible, we assume $R_0 < 1$ in the rest of this section. Let us now consider the admissible behavior of $v(\phi)$ at large ϕ . It is convenient to change the independent variable from ϕ to χ defined by Eq. (A144) and to rewrite Eq. (A141) as

$$\frac{dv}{d\chi} = \frac{1}{n-1} \left[\frac{Q_0}{nQ_1 v^{n-1}} \left(\frac{2}{\sqrt{K}} \right)_{,\phi} - 3\kappa \sqrt{\frac{Q_0}{1-R}} \right]. \quad (\text{A160})$$

For an asymptotically stable solution, we need $v(\phi) \rightarrow 0$ while $v(\phi) > 0$. Therefore, $dv/d\phi$ (and therefore also $dv/d\chi$) must remain negative at large ϕ . Let us examine the condition under which the right-hand side of Eq. (A160) might be negative at large ϕ .

We notice that the first term in the right-hand side of Eq. (A160) contains a negative power of v multiplied by a nonnegative function $d(K^{-1/2})/d\phi$ and a positive constant. This term will diverge to positive infinity as $v \rightarrow 0$ unless $d(K^{-1/2})/d\phi$ tends to zero at large ϕ . On the other hand, the second term,

$$-3\kappa \sqrt{\frac{Q_0}{1-R}}, \quad (\text{A161})$$

tends to a negative constant at large ϕ . Thus, $dv/d\chi$ may become negative at large ϕ only when $d(K^{-1/2})/d\phi$ tends to zero at large ϕ . If $K(\phi)$ is such that $K'K^{-3/2} \rightarrow 0$, then $\chi(\phi) \equiv \int^\phi \sqrt{K(\phi)} d\phi$ diverges at $\phi \rightarrow \infty$; this was already shown in the previous section after Eq. (A129). Let us therefore continue the analysis under the assumptions (A129) and $R_0 < 1$, taking into account that $\chi \rightarrow \infty$ together with $\phi \rightarrow \infty$.

Rewriting Eq. (A142) as

$$\frac{dR}{d\chi} = -\frac{3\kappa}{v} R \sqrt{1-R} \sqrt{Q_0} (w_m + 1) [1 + O(v)], \quad (\text{A162})$$

and noting that $w_m + 1 \neq 0$, we immediately see that $dR/d\chi \rightarrow 0$ can be realized only if $w_m + 1 > 0$ and either

$R_0 = 0$ or $R_0 = 1$, where $R_0 \equiv \lim_{\phi \rightarrow \infty} R(\phi)$. Since we are assuming $R_0 < 1$, the only admissible value is $R_0 = 0$. Therefore, we now look for solutions of Eqs. (A160)–(A162) such that $v(\chi) \rightarrow 0$ and $R(\chi) \rightarrow 0$ as $\chi \rightarrow \infty$ (at the same time as $\phi \rightarrow \infty$).

Computing the limit of Eq. (A160) as $\phi \rightarrow \infty$ and noting that $dv/d\chi \rightarrow 0$ on asymptotically stable solutions, we obtain the condition

$$\lim_{\phi \rightarrow \infty} \frac{Q_0}{nQ_1} \frac{1}{v^{n-1}} \left(\frac{2}{\sqrt{K}} \right)_{,\phi} = 3\kappa \sqrt{Q_0}. \quad (\text{A163})$$

The right-hand side above is a nonzero constant. Therefore it suffices to look for solutions $v(\phi)$ of the form

$$v^{n-1}(\phi) = \frac{\sqrt{Q_0}}{3\kappa n Q_1} \left(\frac{2}{\sqrt{K}} \right)_{,\phi} [1 + A(\phi)], \quad (\text{A164})$$

where $A(\phi)$ is a new unknown function replacing $v(\phi)$. Solutions $v(\phi) \rightarrow 0$ will be asymptotically stable if the general solution for $A(\phi)$ tends to zero as $\phi \rightarrow \infty$. For brevity, we rewrite the ansatz (A164), with the independent variable ϕ expressed through χ , as

$$v(\chi) = [(1 + A(\chi)) W(\chi)]^{\frac{1}{n-1}}, \quad (\text{A165})$$

where $W(\chi)$ is a fixed function defined through

$$W(\chi)|_{\chi=\chi(\phi)} = \frac{\sqrt{Q_0}}{3\kappa n Q_1} \left(\frac{2}{\sqrt{K}} \right)_{,\phi}. \quad (\text{A166})$$

By assumption, we have $W(\chi) \rightarrow 0$ as $\chi \rightarrow \infty$. Substituting the ansatz (A165) into Eqs. (A160)–(A162), we obtain, to the leading order in A and R ,

$$\frac{dA}{d\chi} = -\frac{3\kappa \sqrt{Q_0} \left(\frac{1}{2} R + A \right) + (n-1) \left(W^{\frac{1}{n-1}} \right)_{,\chi}}{W^{\frac{1}{n-1}}}, \quad (\text{A167})$$

$$\frac{dR}{d\chi} = -3\kappa \sqrt{Q_0} (w_m + 1) R W^{-\frac{1}{n-1}}. \quad (\text{A168})$$

Since the equation for R does not contain A , the stability analysis can be performed first for $R(\chi)$ and then for $A(\chi)$ assuming that $R(\chi) \rightarrow 0$. It is convenient to replace the independent variable χ temporarily by

$$\psi(\chi) \equiv \int^\chi W^{-\frac{1}{n-1}}(\chi) d\chi. \quad (\text{A169})$$

Since $W(\chi) \rightarrow 0$ as $\chi \rightarrow \infty$, the new variable ψ grows to infinity together with χ . The new equations for $A(\psi)$ and $R(\psi)$ are

$$\frac{dA}{d\psi} = -3\kappa \sqrt{Q_0} \left(\frac{1}{2} R + A \right) - (n-1) \left(W^{\frac{1}{n-1}} \right)_{,\chi}, \quad (\text{A170})$$

$$\frac{dR}{d\psi} = -3\kappa \sqrt{Q_0} (w_m + 1) R. \quad (\text{A171})$$

It is clear that the general solution for $R(\psi)$ tends to zero if $w_m > -1$. The general solution for $A(\psi)$ is a sum of the general homogeneous solution (which tends to zero) and an inhomogeneous solution. The inhomogeneous terms are proportional to R and $(W^{1/(n-1)})_{,\chi}$, both of which tend to zero at $\chi \rightarrow \infty$ ($\psi \rightarrow \infty$). Therefore the general solution for $v(\phi)$ and $R(\phi)$ is asymptotically stable under the current assumptions.¹¹

8. Slow motion ($v_0 = 0$), marginal cases ($Q(0) = 0$)

Let us now turn to the case $Q(0) = 0$. In this case, we may expand the relevant quantities near $v = 0$ as follows,

$$Q(v) = Q_1 v^n [1 + O(v)], \quad (\text{A172})$$

$$\tilde{\varepsilon}_\phi(v) = (n-1) Q_1 v^n [1 + O(v)], \quad (\text{A173})$$

$$w_\phi(v) = \frac{1}{n-1} [1 + O(v)], \quad (\text{A174})$$

$$c_s^2(v) = \frac{1}{n-1} [1 + O(v)], \quad (\text{A175})$$

where $n \geq 2$ and $Q_1 > 0$. Using these expansions, we rewrite the equations of motion (35)–(36), in the leading order in v , as

$$\frac{dv}{d\phi} = -\frac{v}{n} \frac{K'}{K} - \frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}} \sqrt{\frac{K(\phi)}{1-R}} v^{\frac{n}{2}}, \quad (\text{A176})$$

$$\frac{dR}{d\phi} = -3\kappa R \sqrt{1-R} \sqrt{(n-1) Q_1 K(\phi)} \frac{w_m - w_\phi(v)}{v^{1-n/2}}. \quad (\text{A177})$$

The possible asymptotic values of equation of state parameter $w_\phi(0)$ are $1/(n-1)$ for $n \geq 2$; in particular, we can have $w_\phi(0) = \frac{1}{3}$, mimicking radiation, if $n = 4$. When $w_m = 1/(n-1)$, we may need to expand the term $w_m - w_\phi(v)$ to a higher nonvanishing order in v . For instance, assuming an expansion

$$Q(v) \equiv Q_1 v^n + Q_2 v^{n+p} [1 + O(v)], \quad (\text{A178})$$

where $n \geq 2$ and $p \geq 1$, we find

$$w_\phi(v) = \frac{Q(v)}{vQ'(v) - Q} = \frac{1 + O(v)}{n-1} \left[1 - \frac{pQ_2 v^p}{(n-1)Q_1} \right]. \quad (\text{A179})$$

Let us begin by considering the possible asymptotic value $R_0 = 1$ of $R(\phi)$ at $\phi \rightarrow \infty$; values $R_0 < 1$ will be considered subsequently. In case $R_0 = 1$, we write the ansatz $R(\phi) = 1 - B(\phi)$ and transform Eq. (A177) into

$$\frac{d\sqrt{B}}{d\phi} = \frac{3}{2} \kappa \sqrt{(n-1) Q_1 K(\phi)} v^{\frac{n}{2}-1} (w_m - w_\phi(v)). \quad (\text{A180})$$

The right-hand side of the equation above must be negative to allow $\sqrt{B(\phi)} \rightarrow 0$ at $\phi \rightarrow \infty$. This cannot happen if $w_m - w_\phi(0) > 0$. Thus, the only possibility for the existence of stable solutions is $w_\phi(v) > w_m$ for $v > 0$ (which does not exclude $w_\phi(0) = w_m$). Under the assumption $w_\phi(0) > w_m$, Eqs. (A176) and (A180) can be rewritten (again keeping only the leading-order terms) as

$$\frac{d}{d\phi} \ln(K^{1/n} v) = -\frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}} \sqrt{\frac{K(\phi)}{B(\phi)}} v^{\frac{n}{2}-1}, \quad (\text{A181})$$

$$\frac{d \ln B}{d\phi} = -\frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}} (1 - (n-1)w_m) \sqrt{\frac{K(\phi)}{B(\phi)}} v^{\frac{n}{2}-1}. \quad (\text{A182})$$

In case $w_\phi(0) = w_m$, we assume the expansion (A178) and use Eq. (A179); then Eq. (A182) is replaced by

$$\frac{d \ln B}{d\phi} = \frac{3\kappa p Q_2}{(n-1)^{3/2} \sqrt{Q_1}} \sqrt{\frac{K(\phi)}{B(\phi)}} v^{\frac{n}{2}-1+p}. \quad (\text{A183})$$

As in the case $w_\phi(0) \neq w_m$, stable solutions are possible only if the right-hand side of Eq. (A183) is negative, i.e. if $Q_2 < 0$.

We now need to analyze the solutions of the systems (A181)–(A182) and (A181), (A183) by looking for such $K(\phi)$ that the general solutions $v(\phi)$ and $B(\phi)$ always tend to zero in the two cases.

The general solution of Eqs. (A181)–(A182) can be found by first noticing that

$$\frac{d}{d\phi} \left[\ln(K^{1/n} v) - \frac{\ln B}{1 - (n-1)w_m} \right] = 0. \quad (\text{A184})$$

Hence we may express

$$B(\phi) = C_0 \left[K^{1/n} v \right]^{1 - (n-1)w_m}, \quad (\text{A185})$$

where $C_0 > 0$ is an integration constant. Then we substitute this $B(\phi)$ into Eq. (A181) and obtain the following equation for the auxiliary function $u \equiv K^{1/n} v$,

$$\frac{du}{d\phi} = -F(\phi) u^s, \quad (\text{A186})$$

where we defined the auxiliary constant s and function $F(\phi)$ as

$$s \equiv \frac{n-1}{2} (1 + w_m), \quad (\text{A187})$$

$$F(\phi) \equiv 3\kappa \sqrt{\frac{Q_1}{(n-1)C_0}} K^{1/n}(\phi) > 0. \quad (\text{A188})$$

Since by assumption $w_m < \frac{1}{n-1}$, the possible values of s are $s < \frac{n}{2}$. We are now looking for functions $K(\phi)$ such that both $v = u/F$ and $B \propto u^{n-2s}$ always tend to zero as $\phi \rightarrow \infty$; in other words, we require

$$\lim_{\phi \rightarrow \infty} \frac{u(\phi)}{F(\phi)} = 0, \quad \lim_{\phi \rightarrow \infty} u(\phi) = 0 \quad (\text{A189})$$

¹¹ This is case 10 in Sec. V A.

for the general solution $u(\phi)$. The general solution for $u(\phi)$ can be written as

$$u(\phi) = \begin{cases} \left[C_1 + (s-1) \int^\phi F(\phi) d\phi \right]^{1/(1-s)}, & s \neq 1, \\ \exp \left(C_1 - \int^\phi F(\phi) d\phi \right), & s = 1, \end{cases} \quad (\text{A190})$$

where C_1 is a constant of integration. If $s < 1$, the power $1/(1-s)$ is positive and so the general solution $u(\phi)$ does not tend to zero. If $s \geq 1$, the general solution $u(\phi)$ tends to zero in case $\int^\phi F(\phi) d\phi$ diverges as $\phi \rightarrow \infty$, and does not tend to zero if $\int^\phi F(\phi) d\phi$ converges. Therefore, the only possibility for a stable solution is $s \geq 1$ and $\int^\phi F(\phi) d\phi \rightarrow \infty$ as $\phi \rightarrow \infty$, or equivalently

$$w_m > -\frac{n-3}{n-1}; \quad \lim_{\phi \rightarrow \infty} \int^\phi K^{1/n}(\phi) d\phi = \infty. \quad (\text{A191})$$

It remains to examine the condition $u(\phi)/F(\phi) \rightarrow 0$ under these assumptions.

Since we already have $u \rightarrow 0$, the condition $u/F \rightarrow 0$ holds if $F(\phi)$ approaches a nonzero constant or infinity as $\phi \rightarrow \infty$. However, if

$$\lim_{\phi \rightarrow \infty} F(\phi) = \lim_{\phi \rightarrow \infty} K^{1/n}(\phi) = 0, \quad (\text{A192})$$

the condition $u/F \rightarrow 0$ is a nontrivial additional constraint on the function $K(\phi)$. This constraint can be expressed as a condition on $K(\phi)$ as follows. We find from Eq. (A190) that

$$\frac{u}{F} \propto \begin{cases} \left[F^{s-1} \int^\phi F(\phi) d\phi \right]^{-1/(s-1)}, & s > 1, \\ \exp \left(-\ln F - \int^\phi F(\phi) d\phi \right), & s = 1. \end{cases} \quad (\text{A193})$$

The condition $u/F \rightarrow 0$ is then equivalent to

$$\begin{aligned} \lim_{\phi \rightarrow \infty} F^{s-1} \int^\phi F(\phi) d\phi &= \infty, \quad s > 1; \quad (\text{A194}) \\ \lim_{\phi \rightarrow \infty} \left(\ln F + \int^\phi F(\phi) d\phi \right) &= \infty, \quad s = 1. \quad (\text{A195}) \end{aligned}$$

We note that the left-hand sides in Eqs. (A194)–(A195) depend monotonically on the growth of $F(\phi)$; more precisely, the terms under the limits become larger when we choose a faster-growing or slower-decaying function $F(\phi)$. Thus, it is clear that the conditions (A194)–(A195) will hold if $F(\phi)$ decays sufficiently slowly as $\phi \rightarrow \infty$ (or grows, but this case was already considered). With some choices of $F(\phi) = F_0(\phi)$, the limits in Eqs. (A194)–(A195) will be finite nonzero constants. We can easily determine such $F_0(\phi)$,

$$F_0(\phi) \propto \phi^{-1/s}, \quad s \geq 1, \quad \phi \rightarrow \infty. \quad (\text{A196})$$

Hence, the limits (A194)–(A195) will be infinite when $F(\phi)$ decays slower than $\phi^{-1/s}$. The corresponding condition for $K(\phi)$ can be written as

$$\lim_{\phi \rightarrow \infty} \phi^{n/s} K(\phi) = \infty. \quad (\text{A197})$$

We can make this argument more rigorous by assuming the ansatz

$$K(\phi) = \phi^{-n/s} K_1(\phi), \quad (\text{A198})$$

where $K_1(\phi) > 0$ is an auxiliary function. Note that the function $F(\phi)$ is related to $K(\phi)$ by Eq. (A188), which contains an arbitrary integration constant $C_0 > 0$. Thus we may write

$$F(\phi) = C_1 \phi^{-1/s} K_1^{1/n}(\phi), \quad (\text{A199})$$

where $C_1 > 0$ is an arbitrary constant. If

$$\lim_{\phi \rightarrow \infty} K_1(\phi) = \infty, \quad (\text{A200})$$

it means that $K_1(\phi)$ is larger than any constant at sufficiently large ϕ . Then we obtain lower bounds (for arbitrary constant $C_2 > 0$)

$$\int^\phi F(\phi) d\phi > C_2 \phi^{1-1/s}, \quad s > 1; \quad (\text{A201})$$

$$\int^\phi F(\phi) d\phi > C_2 \ln \phi, \quad s = 1, \quad (\text{A202})$$

and the conditions (A194)–(A195) hold. On the other hand, if

$$\lim_{\phi \rightarrow \infty} K_1(\phi) \equiv K_1^{(0)} < \infty, \quad (\text{A203})$$

we find

$$\int^\phi F(\phi) d\phi \approx C_2 \phi^{1-1/s}, \quad s > 1; \quad (\text{A204})$$

$$\int^\phi F(\phi) d\phi \approx C_2 \ln \phi, \quad s = 1, \quad (\text{A205})$$

where $C_2 > 0$ is an arbitrary constant. In that case, the conditions (A194)–(A195) cannot hold for arbitrary C_2 . Therefore, the condition (A197) is necessary and sufficient for Eqs. (A194)–(A195) to hold.

We conclude that an asymptotically stable solution exists for $v_0 = 0$, $Q(0) = 0$ with the expansion (A178), when $R_0 = 1$, $w_m < \frac{1}{n-1}$, and the conditions (A191) and (A197) hold. We note that for $n = 2$ the condition $w_m < \frac{1}{n-1}$ contradicts the first condition in Eq. (A191), so admissible solutions exist only for $n > 2$.¹²

The remaining case requires the analysis of Eqs. (A181)–(A183). The general solution of these equations cannot be obtained in closed form; however, we only need to analyze the asymptotic behavior at $\phi \rightarrow \infty$. So we will estimate the relative magnitude

¹² This is case 6 in Sec. V A.

of different terms in these equations. Let us rewrite Eqs. (A181)–(A183) as

$$\frac{d \ln v}{d\phi} = -\frac{1}{n} \frac{K'}{K} - \tilde{Q}_1 \frac{K^{1/2}}{\sqrt{B}} v^{n/2-1}, \quad (\text{A206})$$

$$\frac{d\sqrt{B}}{d\phi} = -\tilde{Q}_2 K^{1/2} v^{n/2-1+p}, \quad (\text{A207})$$

where the auxiliary positive constants

$$\tilde{Q}_1 \equiv \frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}}, \quad \tilde{Q}_2 \equiv -\frac{3\kappa p Q_2}{(n-1)^{3/2} \sqrt{Q_1}} \quad (\text{A208})$$

were introduced for brevity. (Positivity of these constants is clearly necessary for the existence of asymptotically stable solutions.) Suppose that $v(\phi)$ and $B(\phi)$ are decaying solutions of Eqs. (A206)–(A207), and let us compare the magnitude of the terms in the right-hand side of Eq. (A206) in the limit $\phi \rightarrow \infty$. There are only three possibilities: the first term dominates; the two terms have the same order; or the second term dominates. In other words, the limit of the ratio of the second term to the first,

$$q \equiv \lim_{\phi \rightarrow \infty} \frac{\tilde{Q}_1 K^{3/2} v^{n/2-1}}{K' \sqrt{B}}, \quad (\text{A209})$$

must be either zero, or finite but nonzero, or infinite. The value of q must be the same for every decaying solution $\{v(\phi), B(\phi)\}$ except perhaps for a discrete subset of solutions, which we may ignore for the purposes of stability analysis. In each of the three cases, Eqs. (A206)–(A207) are simplified and become amenable to asymptotic analysis in the limit $\phi \rightarrow \infty$. We will now consider these three possible values of q in turn.

If $q = 0$, we have at large ϕ

$$\frac{1}{n} \frac{K'}{K} \gg \tilde{Q}_1 \frac{K^{1/2}}{\sqrt{B}} v^{n/2-1}, \quad (\text{A210})$$

and thus only the first term is left in Eq. (A206),

$$\frac{d \ln v}{d\phi} \approx -\frac{1}{n} \frac{K'}{K} \Rightarrow v \propto K^{-1/n}. \quad (\text{A211})$$

Decaying solutions have $v(\phi) \rightarrow 0$; so a necessary condition is $K^{-1/n}(\phi) \rightarrow 0$ at $\phi \rightarrow \infty$. With this $v(\phi)$, the condition (A210) becomes

$$(K^{-1/n})' \gg \frac{\tilde{Q}_1}{\sqrt{B}}. \quad (\text{A212})$$

However, this condition cannot be satisfied, since the left-hand side tends to zero at large ϕ while the right-hand side tends to infinity because $B \rightarrow 0$. Thus, decaying solutions $v(\phi), B(\phi)$ are impossible with $q = 0$.

If $q \neq 0$ and $|q| < \infty$, we consider q as an unknown constant that possibly depends on the solutions $v(\phi)$ and

$B(\phi)$. At large ϕ , we have

$$q \frac{K'}{K} \approx \tilde{Q}_1 \frac{1}{\sqrt{B}} v^{n/2-1} K^{1/2}, \quad (\text{A213})$$

$$\frac{d \ln v}{d\phi} \approx -\frac{K'}{K} \left(\frac{1}{n} + q \right). \quad (\text{A214})$$

Therefore,

$$v(\phi) \propto K^{-q-1/n}(\phi) \rightarrow 0 \quad (\text{A215})$$

since we need a decaying solution $v(\phi)$. With this $v(\phi)$, Eq. (A213) yields

$$\sqrt{B} \approx \frac{\tilde{Q}_1}{q} \frac{K^{1-(n/2-1)q+1/n}}{K'}. \quad (\text{A216})$$

For a decaying solution $B(\phi) \rightarrow 0$, we thus must have

$$\frac{1}{\sqrt{B}} \propto \frac{d}{d\phi} \left[K^{(n/2-1)q-1/n} \right] \rightarrow \infty \text{ as } \phi \rightarrow \infty, \quad (\text{A217})$$

and in particular

$$\left(\frac{n}{2} - 1 \right) q - \frac{1}{n} \neq 0. \quad (\text{A218})$$

Substituting the expressions for $v(\phi)$ and $\sqrt{B(\phi)}$ into Eq. (A207), we find

$$\begin{aligned} \frac{d\sqrt{B}}{d\phi} &= -\tilde{Q}_2 K^{\frac{1}{2}-(\frac{n}{2}-1)p} (q+\frac{1}{n}) \\ &= \frac{d}{d\phi} \frac{\tilde{Q}_1}{q} \frac{K^{1-(\frac{n}{2}-1)q+\frac{1}{n}}}{K'} \\ &= \frac{\tilde{Q}_1}{q} K^{-(\frac{n}{2}-1)q+\frac{1}{n}} \left[1 - \left(\frac{n}{2} - 1 \right) q + \frac{1}{n} - \frac{KK''}{K'^2} \right]. \end{aligned} \quad (\text{A219})$$

This is now a closed equation for $K(\phi)$, which we may rewrite as

$$\begin{aligned} \left(\frac{K}{K'} \right)' &= 1 - \frac{KK''}{K'^2} \\ &= \left[\left(\frac{n}{2} - 1 \right) q - \frac{1}{n} \right] - \frac{q\tilde{Q}_2}{\tilde{Q}_1} K^{-p(q+\frac{1}{n})}. \end{aligned} \quad (\text{A220})$$

Due to the conditions (A215), (A218), and since $p > 0$, the right-hand side above tends to a nonzero limit as $\phi \rightarrow \infty$, namely

$$\left(\frac{K}{K'} \right)' \approx \left(\frac{n}{2} - 1 \right) q - \frac{1}{n} \equiv \frac{1}{\alpha} \neq 0. \quad (\text{A221})$$

It follows that the only admissible form of the function $K(\phi)$ is

$$K(\phi) \propto \phi^\alpha, \quad \phi \rightarrow \infty. \quad (\text{A222})$$

However, this expression does not satisfy Eq. (A217). Therefore, asymptotically stable solutions are impossible.

In the last case, $q = \infty$, we may disregard the first term in Eq. (A206) and obtain

$$\frac{d \ln v}{d \phi} \approx -\tilde{Q}_1 \frac{K^{1/2}}{\sqrt{B}} v^{n/2-1}. \quad (\text{A223})$$

Then we can rewrite Eq. (A207) as

$$\begin{aligned} \frac{d \ln \sqrt{B}}{d \phi} &= -\tilde{Q}_2 \frac{K^{1/2}}{\sqrt{B}} v^{n/2-1+p} = \frac{\tilde{Q}_2}{\tilde{Q}_1} v^p \frac{d \ln v}{d \phi} \\ &= \frac{\tilde{Q}_2}{p \tilde{Q}_1} \frac{d}{d \phi} v^p. \end{aligned} \quad (\text{A224})$$

This relationship between B and v can be integrated and yields

$$\sqrt{B} = \exp \left[C_1 + \frac{\tilde{Q}_2}{p \tilde{Q}_1} v^p \right], \quad (\text{A225})$$

where C_1 is a constant of integration. It follows that it is impossible to find simultaneously decaying solutions $v(\phi) \rightarrow 0$ and $B(\phi) \rightarrow 0$ at $\phi \rightarrow \infty$.

This concludes the consideration of the case $R_0 = 1$ and $w_m = \frac{1}{n-1}$, in which case there are no asymptotically stable solutions.

We now turn to the analysis of the case $R_0 < 1$. We first note that the leading terms of Eq. (A176) do not contain R when $R \rightarrow R_0 < 1$. Therefore the stability analysis can be performed for $R(\phi)$ and $v(\phi)$ separately. Using the ansatz $R(\phi) = R_0 + B(\phi)$ and assuming a fixed solution $v(\phi)$, we find that the right-hand side of Eq. (A177) is independent of $B(\phi)$ if $0 < R_0 < 1$. Therefore, general solutions $B(\phi)$ will not approach zero in case $R_0 \neq 0$. It remains to look for asymptotically stable solutions $v(\phi)$ and $R(\phi)$ in case $R_0 = 0$.

In case $R_0 = 0$, we begin by analyzing the asymptotic behavior of $v(\phi)$. Rewriting Eq. (A176) as

$$\frac{du}{d\phi} = -\frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}} u^{\frac{n}{2}} K^{1/n}, \quad u \equiv K^{1/n} v, \quad (\text{A226})$$

we find the approximate general solutions (valid only for large ϕ)

$$u(\phi) = \exp \left[-3\kappa\sqrt{Q_1} \int_{\phi_0}^{\phi} K^{1/2} d\phi \right], \quad n = 2, \quad (\text{A227})$$

$$u(\phi) = \left[\frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}} \int_{\phi_0}^{\phi} K^{1/n} d\phi \right]^{-\frac{2}{n-2}}, \quad n > 2, \quad (\text{A228})$$

where ϕ_0 is an integration constant. The general solution $v = K^{-1/n} u$ should tend to zero as $\phi \rightarrow \infty$. We note that Eq. (A226) is similar to Eq. (A186) after the replacements

$$F(\phi) \equiv \frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}} K^{1/n}(\phi), \quad s \equiv \frac{n}{2}. \quad (\text{A229})$$

Therefore, we may use the conclusion obtained after Eq. (A190), with the caveat that $F(\phi)$ is presently related to $K(\phi)$ uniquely, without an arbitrary proportionality factor. This was used to exclude the boundary case (A203), which is presently still allowed. Thus the condition (A197) obtained above,

$$\lim_{\phi \rightarrow \infty} \phi^{n/s} K(\phi) = \lim_{\phi \rightarrow \infty} \phi^2 K(\phi) = \infty, \quad (\text{A230})$$

is now merely a *sufficient* condition for the stability of the general solution $v(\phi)$. In the boundary case,

$$\lim_{\phi \rightarrow \infty} \phi^2 K(\phi) \equiv K_0, \quad 0 < K_0 < \infty, \quad (\text{A231})$$

we find

$$v(\phi) \propto \exp \left[\left(1 - 3\kappa\sqrt{Q_1 K_0} \right) \ln \phi \right], \quad n = 2, \quad (\text{A232})$$

$$v(\phi) \approx \text{const}, \quad n > 2. \quad (\text{A233})$$

Thus, the case (A231) yields a stable solution for $v(\phi)$ when $n = 2$ and $3\kappa\sqrt{Q_1 K_0} > 1$. (The possibility $3\kappa\sqrt{Q_1 K_0} = 1$ is unphysical because it requires an infinitely precise fine-tuning of the parameters in the field Lagrangian.) Thus a sharp condition for the asymptotic stability of $v(\phi)$ is

$$K(\phi) \geq \frac{1}{9\kappa^2 Q_1} \phi^{-2} \quad \text{at } \phi \rightarrow \infty, \quad n = 2; \quad (\text{A234})$$

$$\lim_{\phi \rightarrow \infty} \phi^2 K(\phi) = \infty, \quad n > 2. \quad (\text{A235})$$

A weaker necessary condition is

$$\int_{\phi_0}^{\infty} K^{1/n}(\phi) d\phi = \infty. \quad (\text{A236})$$

Let us now consider the stability of the general solution for $R(\phi)$. It follows from Eq. (A177) that

$$\frac{d \ln B}{d \phi} = -3\kappa\sqrt{(n-1)Q_1 K(\phi)} v^{\frac{n}{2}-1} (w_m - w_\phi(v)). \quad (\text{A237})$$

This equation integrates to

$$B(\phi) = B_0 \exp \left[-\text{const} \cdot \int_{\phi_0}^{\phi} (w_m - w_\phi(v)) K^{\frac{1}{2}} v^{\frac{n}{2}-1} d\phi \right], \quad (\text{A238})$$

where B_0 is an integration constant. The general solution for $B(\phi)$ will tend to zero as long as the integral in Eq. (A238) diverges to a positive infinity at $\phi \rightarrow \infty$,

$$\int_{\phi_0}^{\infty} (w_m - w_\phi(v)) K^{\frac{1}{2}} v^{\frac{n}{2}-1} d\phi = \infty. \quad (\text{A239})$$

A necessary condition for that is $w_m \geq \frac{1}{n-1}$. Precise constraints on $K(\phi)$ for Eq. (A239) can be obtained by considering the cases $n = 2$, $n \neq 2$, $w_m = \frac{1}{n-1}$, and $w_m > \frac{1}{n-1}$ separately.

If $w_m > \frac{1}{n-1}$, the condition (A239) holds when

$$\int^\infty K^{\frac{1}{2}} v^{\frac{n}{2}-1} d\phi = \infty. \quad (\text{A240})$$

If $n = 2$, the above integral diverges due to the necessary condition (A236). If $n > 2$, we use the solution (A228), where $u = K^{1/n}v$, to obtain

$$\int^\infty K^{\frac{1}{2}} v^{\frac{n}{2}-1} d\phi = \int^\infty d\phi \left[\int_{\phi_0}^\phi K^{1/n}(\phi_1) d\phi_1 \right]^{-1} K^{1/n}(\phi). \quad (\text{A241})$$

Temporarily introducing the auxiliary function

$$I(\phi) \equiv \int^\phi K^{1/n}(\phi) d\phi, \quad (\text{A242})$$

we note that $\lim_{\phi \rightarrow \infty} I(\phi) = \infty$ by Eq. (A236). Therefore we express Eq. (A241) through $I(\phi)$ and obtain

$$\begin{aligned} \int^\infty K^{\frac{1}{2}} v^{\frac{n}{2}-1} d\phi &= \int^\infty \frac{I'(\phi)}{I(\phi)} d\phi \\ &= \lim_{\phi \rightarrow \infty} \ln I(\phi) + \text{const} = \infty. \end{aligned} \quad (\text{A243})$$

Therefore, the general solution $B(\phi)$ tends to zero with $w_m > \frac{1}{n-1}$ for any $n \geq 2$ under the condition (A236).

When $w_m = \frac{1}{n-1}$, it follows from Eq. (A178) that the integrand in Eq. (A238) acquires an additional factor proportional to v^p , where $p \geq 1$. Therefore the general solution for $B(\phi)$ will tend to zero only if

$$\int^\infty K^{1/2} v^{n/2-1+p} d\phi = \infty, \quad (\text{A244})$$

where we need to substitute $v(\phi) = K^{-1/n}u(\phi)$ and $u(\phi)$ as given by Eqs. (A227)–(A228).

Consider first the case $n = 2$; we will now show that the condition (A244) is incompatible with the earlier condition (A234). Using the solution (A227), we can rewrite the condition (A244) as

$$\int^\infty d\phi K^{(1-p)/2} \exp \left[-3p\kappa\sqrt{Q_1} \int_{\phi_0}^\phi K^{1/2} d\phi_1 \right] = \infty. \quad (\text{A245})$$

By the condition (A234), we have

$$K^{(1-p)/2} < \text{const} \cdot \phi^{p-1}, \quad (\text{A246})$$

$$\int_{\phi_0}^\phi K^{1/2}(\phi_1) d\phi_1 > \sqrt{K_0} \ln \phi + \text{const}. \quad (\text{A247})$$

Therefore the integral in Eq. (A245) is bounded from above by

$$\text{const} \int^\infty d\phi \phi^{p-1-3p\kappa\sqrt{Q_1K_0}} = \text{const} \int^\infty d\phi \phi^{-1-\alpha} < \infty, \quad (\text{A248})$$

where we temporarily denoted

$$\alpha \equiv \left(3\kappa\sqrt{Q_1K_0} - 1 \right) p > 0, \quad (\text{A249})$$

and so the condition (A245) cannot hold.

It remains to consider the case $w_m = \frac{1}{n-1}$ and $n > 2$. Using Eq. (A228), we rewrite the condition (A244) as

$$\int^\infty K^{\frac{1-p}{n}}(\phi) \left[\int_{\phi_0}^\phi K^{1/n}(\phi_1) d\phi_1 \right]^{-\frac{2p}{n-2}-1} d\phi = \infty. \quad (\text{A250})$$

According to Eq. (A235), we must have

$$K(\phi) > C_0 \phi^{-2} \quad (\text{A251})$$

for any $C_0 > 0$ at large enough ϕ ; thus $K(\phi)$ should decay slower than ϕ^{-2} . However, it is straightforward to verify that a power-law behavior

$$K(\phi) \propto \phi^{-2+\delta}, \quad \phi \rightarrow \infty, \quad \delta > 0, \quad (\text{A252})$$

yields a convergent integral in Eq. (A250). Therefore, the only possibility of having an asymptotically stable solution is to choose $K(\phi)$ such that it decays slower than ϕ^{-2} but faster than $\phi^{-2+\delta}$ for any $\delta > 0$. An example of an admissible choice of $K(\phi)$ is

$$K(\phi) \propto \phi^{-2} (\ln \phi)^\alpha, \quad \alpha > 0. \quad (\text{A253})$$

With this $K(\phi)$, we obtain the following asymptotic estimate at large ϕ ,

$$\int_{\phi_0}^\phi K^{1/n}(\phi_1) d\phi_1 \propto \text{const} \cdot \phi^{1-2/n} (\ln \phi)^{\alpha/n}, \quad (\text{A254})$$

and so the integral (A250) becomes, after some algebra,

$$\text{const} \int^\infty \phi^{-1} (\ln \phi)^{-\alpha \frac{p}{n-2}} d\phi = \infty \quad \text{if} \quad \alpha \frac{p}{n-2} \leq 1. \quad (\text{A255})$$

Since the convergence of the integral in Eq. (A250) monotonically depends on the growth properties of the function $K(\phi)$, it is clear that the condition (A250) will also hold for functions $K(\phi)$ satisfying Eq. (A253) but growing slower than those given in Eq. (A253). However, the condition (A250) may not hold for $K(\phi)$ growing faster than those in Eq. (A253).

To investigate the admissible class of functions $K(\phi)$ more precisely, let us use the ansatz

$$K(\phi) = \phi^{-2} K_0(\phi), \quad (\text{A256})$$

where $K_0(\phi)$ is a function growing slower than any power of ϕ . Then we have an asymptotic estimate (for $n > 2$)

$$\int_{\phi_0}^\phi K^{1/n}(\phi_1) d\phi_1 \approx \text{const} \cdot \phi^{1-2/n} (K_0(\phi))^{1/n}, \quad (\text{A257})$$

and we can rewrite Eq. (A244) as

$$\int_{\phi_0}^{\infty} K^{\frac{1-p}{n}}(\phi) \left[\int_{\phi_0}^{\phi} K^{1/n}(\phi_1) d\phi_1 \right]^{-\frac{2p}{n-2}-1} d\phi \\ = \text{const} \cdot \int_{\phi_0}^{\infty} \phi^{-1} [K_0(\phi)]^{-\frac{p}{n-2}} d\phi = \infty. \quad (\text{A258})$$

Substituting $K_0 = \phi^2 K$ into Eq. (A258), we find that the conditions (A235) and (A250) are equivalent to

$$\lim_{\phi \rightarrow \infty} \phi^2 K(\phi) = \infty, \quad \int_{\phi_0}^{\infty} \phi^{-1-\frac{2p}{n-2}} [K(\phi)]^{-\frac{p}{n-2}} d\phi = \infty. \quad (\text{A259})$$

The condition (A235) guarantees the stability of $v(\phi)$, while Eq. (A250) guarantees the stability of $B(\phi)$. Therefore, Eq. (A259) is a sharp (necessary and sufficient) condition for the stability of the solution $\{v, B\}$.

A sufficient (but not a necessary) condition for the divergence of the integral in Eq. (A258) is

$$\lim_{\phi \rightarrow \infty} (\ln \phi)^{-\frac{n-2}{p}} K_0(\phi) < \infty. \quad (\text{A260})$$

The corresponding sufficient condition for $K(\phi)$ is

$$\lim_{\phi \rightarrow \infty} \phi^2 K(\phi) = \infty, \quad \lim_{\phi \rightarrow \infty} \phi^2 (\ln \phi)^{-\frac{n-2}{p}} K(\phi) < \infty. \quad (\text{A261})$$

¹³ This is case 6 in Sec. V A.

¹⁴ This is case 7 in Sec. V A.

¹⁵ This is case 11 in Sec. V A.

¹⁶ This is case 12 in Sec. V A.

The sharp condition (A259) cannot be restated in terms of the asymptotic behavior of $K(\phi)$ at $\phi \rightarrow \infty$, but of course one can check whether Eq. (A259) holds for a given $K(\phi)$. The condition (A259) specifies a rather narrow class of functions; however, we strive for generality and avoid prejudice regarding the possible Lagrangians.

In this section we have shown that asymptotically stable solutions exist with $v_0 = 0$ and $Q(0) = 0$ only in the following cases: (a) Asymptotic value $R_0 = 1$. Expansion (A172) holds with $Q_1 > 0$, determining the value of n , which should be $n > 2$; $-\frac{n-3}{n-1} < w_m < \frac{1}{n-1}$ according to Eq. (A191); and $K(\phi)$ satisfies Eq. (A197), where s is defined by Eq. (A187).¹³ There are no stable solutions when $w_m = \frac{1}{n-1}$ and expansion (A178) holds. (b) Asymptotic value $R_0 = 0$. Expansion (A172) holds with $Q_1 > 0$, determining the value of $n \geq 2$; either $n = 2$, $w_m > 1$, and $K(\phi)$ satisfies Eq. (A234)¹⁴ or Eq. (A235),¹⁵ or $n > 2$, $w_m = \frac{1}{n-1}$, and $K(\phi)$ satisfies Eq. (A259).¹⁶

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